STATICS AND THE DYNAMICS
OF A PARTICLE

THEORETICAL MECHANICS

STATICS AND THE DYNAMICS OF A PARTICLE

BY

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FIRST EDITION
SECOND IMPRESSION

McGRAW-HILL BOOK COMPANY, INC.
NEW YORK: 370 SEVENTH AVENUE
LONDON: 6 & 8 BOUVERIE ST.; E. C. 4
1927

PRINTED IN THE UNITED STATES OF AMERICA

THE MAPLE PRESS COMPANY, YORK, PA.

PREFACE

The present volume is designed as a textbook in mechanics for the use of students in our colleges and universities, particularly those students who are interested in astronomy, physics, or mathematics. It is not designed primarily for the student of engineering, particularly those who are interested only in the technical aspect of engineering; but it should make an appeal to that relatively small group of engineers who wish to extend their knowledge beyond the mere rule of thumb into the deeper foundations and developments of general theory.

The treatment of the subject is elementary in the sense that it begins at the beginning by developing the concepts and postulates which are peculiar to mechanics. The chapter devoted to vectors requires a knowledge of geometry only. Velocity, acceleration, and force require the notion of a limit, and work that of a definite integral. While this is true, an extended knowledge of the calculus is not required in the first part of the book. The second part, on statics, draws somewhat more heavily on the calculus, while the third part, on the dynamics of a particle, requires a thorough working knowledge of it. It is evident, therefore, that if the student is not already familiar with calculus, a study of this volume must be supplemented by studies in pure mathematics. The two studies can go on together, however, and it is even desirable that they should, for progress in both subjects will be slower, and more time will be allowed for a proper assimilation of the ideas which have been received.

Mechanics is not an easy subject: not only is it not easy for the student, it has proven to be a difficult subject for the entire human race, as is evidenced by the fact that it came into existence two thousand years later than its allied subject, geometry. Philosophers are not yet satisfied with its foundations, and doubtless will not be for some time to come. The difficulties of the subject make many problems desirable, and many problems have been furnished in the text. Familiarity is gained only by much practice, and much material for this practice is necessary.

Inasmuch as students are rarely certain of their results, the answers are given to all of the problems.

An adequate treatment of the entire subject of mechanics cannot be given in a single volume. The present volume contains those subjects which are usually taught in our colleges and universities, namely statics, and the dynamics of a particle. It is the intention of the author to deal with the theory of the potential and the dynamics of rigid, elastic, and fluid bodies in future volumes.

The author's colleague, Dr. Walter Bartky, has very kindly assisted in reading the final proofs.

W. D. MacMillan.

The University of Chicago,
April, 1927.

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STATICS AND THE DYNAMICS OF A PARTICLE

PART I

THE FUNDAMENTAL CONCEPTS OF MECHANICS

CHAPTER I

VECTORS

1. Directed Quantities.—Certain quantities occur in nature which are not completely characterized by magnitude alone. When speaking of four dollars or of ten tons, the quantities are completely characterized by the numbers *four* and *ten*. But in speaking of the wind, it is not sufficient to say that the wind is blowing six miles per hour, for the direction from which the wind is blowing may be more important than its rate. It would be quite sufficient, however, to say that the wind was blowing six miles per hour from the northeast (Fig. 1*a*) or nine miles per hour from the west (Fig. 1*b*). Forces and accelerations also can be completely characterized only by giving both their magnitudes and directions. Quantities of this kind can be represented geometrically by an arrow, or a directed segment of a straight line, the length of the arrow being equal (on a suitable scale) to the numerical value of the quantity in question, while the direction of the arrow is, of course, the same as the direction of the quantity which it represents.

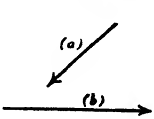


FIG. 1.

2. Definition of a Scalar.—A *scalar* is a quantity which possesses magnitude but not direction. It can be measured by means of a suitable scale. The ordinary numbers of arithmetic or algebra are typical scalars. They are measures of length, time, weight, etc.

3. Definition of a Vector.—A *vector* is a directed quantity which combines with other directed quantities of the same kind in accordance with the parallelogram law. Its position is not material, so that it can be moved about freely provided its length and direction are not altered.

Velocities, accelerations, and forces are typical vectors, as will be shown later in the proper place; but a directed quantity is not necessarily a vector.

4. The Parallelogram Law.—If two directed quantities of the same kind taken together are equivalent to a single directed quantity of the same kind which in magnitude and direction is the diagonal of the parallelogram of which the first two quantities form two of the sides, then the two quantities are said to combine in accordance with the parallelogram law. Thus, in Fig. 2, if the directed quantities $\overrightarrow{AB}$ and $\overrightarrow{AD}$ taken together are equivalent to $\overrightarrow{AC}$, which is the diagonal of the parallelogram formed on the two sides AB and AD, then $\overrightarrow{AB}$ and $\overrightarrow{AD}$ are directed quantities which obey the parallelogram law and are therefore vectors.

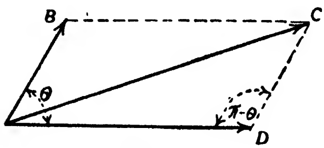


Fig. 2.

If the lengths of $\overrightarrow{AB}$, $\overrightarrow{AD}$, and $\overrightarrow{AC}$ are b, d, and c, respectively, and θ is the angle between $\overrightarrow{AB}$ and $\overrightarrow{AD}$, then from the cosine law of trigonometry applied to the triangle ACD it is found that

$$c^2 = b^2 + d^2 + 2bd \cos \theta. \tag{1}$$

The vector $\overrightarrow{AC}$ is called the *resultant* of $\overrightarrow{AB}$ and $\overrightarrow{AD}$.

5. Composition and Resolution of Vectors.—The process of combining two or more vectors into a single one by means of the parallelogram law is spoken of as the *composition of vectors.* It is readily verified that the result of the composition is quite independent of the order in which the vectors are combined.

Conversely, a single vector can be resolved into two or more components, and this process is called the *resolution of vectors.* Any set of vectors, no two of which have the same direction,

which taken together are equivalent to a single vector is called the components of the single vector. Thus, in Fig. 3, $\overrightarrow{AC}$ and $\overrightarrow{AD}$ are components of $\overrightarrow{AB}$; but so also are $\overrightarrow{AE}$ and $\overrightarrow{AF}$ components of $\overrightarrow{AB}$. If the directions of the components are specified, the resolution is unique, but if nothing at all is specified a vector can be resolved into components in an unlimited number of ways.

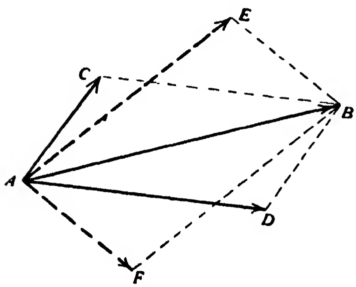


Fig. 3.

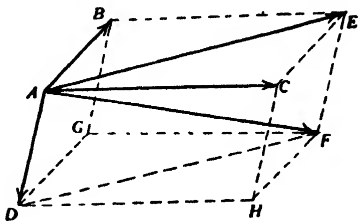


Fig. 4.

6. Vectors in Three Dimensions.—Vectors which lie in three dimensions can be combined into single vectors just as they are in two dimensions, for the resultant of any two vectors can be combined with a third vector even though the third vector does not lie in the same plane as the first two. The resultant of the first three vectors can be combined with a fourth vector, and so on. The number of the dimensions is not material.

The resultant of three vectors which do not lie in the same plane is the diagonal of the parallelepiped of which the three vectors form three of the edges. Let $\overrightarrow{AB}$, $\overrightarrow{AC}$, and $\overrightarrow{AD}$ (Fig. 4) be three vectors not lying in the same plane, and let $ABEC$-$DGFH$ be the parallelepiped constructed on these three vectors. Then the resultant of $\overrightarrow{AB}$, $\overrightarrow{AC}$, and $\overrightarrow{AD}$ is $\overrightarrow{AF}$, for the resultant of $\overrightarrow{AB}$ and $\overrightarrow{AC}$ is $\overrightarrow{AE}$, and the resultant of $\overrightarrow{AE}$ and $\overrightarrow{AD}$ is $\overrightarrow{AF}$.

7. Miscellaneous Properties of Vectors.—A vector can vanish only if its magnitude vanishes, and in this event direction ceases to have any meaning.

Two vectors are equal if, and only if, they have the same magnitude and the same direction.

Since position is not material, a vector is completely specified by giving its projections upon three intersecting axes which have different directions, provided the three axes do not lie in the same plane. If the end points of a vector are the points x_1, y_1, z_1 and x_2, y_2, z_2, respectively, the projections of the vector upon the axes of reference are

$$x_2 - x_1, \qquad y_2 - y_1, \qquad z_2 - z_1,$$

and these three numbers determine the vector.

A vector is specified in polar coordinates by giving its length r and the polar angles of its direction φ and θ.

If $\overrightarrow{AB}$ is a vector, the point A is called its *origin* and the point B is called its *terminus.*

8. Example. Displacements are Vectors.—Perhaps the simplest example of a vector is the displacement of a point. If a point P moves from the position A to the position B, the length $\overrightarrow{AB}$ is the magnitude of the displacement, and its direction is from the origin A to the terminus B.

That such a displacement can be represented by an arrow is evident. To prove that it is a vector it is necessary to show that two such displacements are equivalent to a single displacement which can be obtained by taking the diagonal of the parallelogram of which the two given displacements form two of the sides.

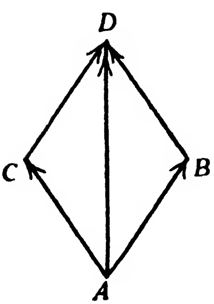


Fig. 5.

In Fig. 5, let $\overrightarrow{AB}$ and $\overrightarrow{AC}$ be two displacements. Let the point P move first from A to B and then be displaced from B by an amount which is equal in magnitude and direction to $\overrightarrow{AC}$. It arrives then at the point D. Or, the order of the displacements can be reversed by letting the point move first from A to C and then be displaced from C by an amount which is equal in magnitude and direction to $\overrightarrow{AB}$. It arrives again at the same point D, since $ABDC$ is a parallelogram. But the point P could be moved from A to D by a single displacement $\overrightarrow{AD}$, and furthermore $\overrightarrow{AD}$

is the diagonal of the parallelogram of which $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are two of the sides.

Hence, the displacement of a point is a vector, since it satisfies all the requirements of the definition.

9. Vector Notation.—The symbol $\overrightarrow{AB}$ which has been used to denote a vector is convenient when it is desired to indicate the origin and terminus of a vector. But since a vector is essentially a single concept, independent of position, it is desirable to represent vectors by means of single letters. Since, however, they differ in their nature from ordinary magnitudes, it is customary to represent them by **bold-face** type in order to call attention to their vector character, *italic* type being used for scalar magnitudes. Thus, **A** is a vector and A is its length, or numerical value; **a** is a vector and a is its length, etc. This practice will be followed throughout the book whenever vector notation is in evidence, as it defines once for all the scalar magnitude of a vector.

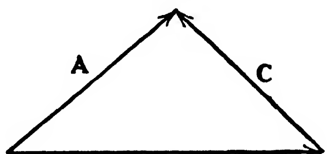


FIG. 6.

10. The Vector Sum.—In Fig. 6, the vectors **B** and **C** taken together are equivalent to the vector **A**. This suggests the notation

$$\mathbf{A} = \mathbf{B} + \mathbf{C}, \tag{1}$$

where the + sign means the vector sum, *i.e.*, compounded according to the parallelogram law, and not according to numerical addition, which would have no meaning. No confusion arises from this use of the addition symbol as the vector notation itself indicates the vector character of the sum.

With such a notation, it would be desired also to have

$$\mathbf{B} + (-\mathbf{B}) = 0,$$

so that $-\mathbf{B}$ would have to be interpreted as differing from **B** only in that its direction is exactly opposite to that of **B**. If, in Fig. 6, **A** is imagined to diminish in length, the triangle remaining always closed, it is evident when **A** vanished that $\mathbf{C} = -\mathbf{B}$, and then Eq. (1) becomes

$$\mathbf{B} + (-\mathbf{B}) = 0.$$

11. The Vector Difference.—In Fig. 7, the vectors **A, B,** and **C** form a closed triangle, so that

$$\mathbf{A} = \mathbf{B} + \mathbf{C} \tag{1}$$

With the same origin as **A**, draw the vector $-\mathbf{C}$. On compounding the vectors **A** and $-\mathbf{C}$, that is, taking their vector sum, there results

$$\mathbf{A} - \mathbf{C} = \mathbf{B};$$

but this is just what would have been obtained had **C** been subtracted from both sides of Eq. (1) just as in ordinary algebra.

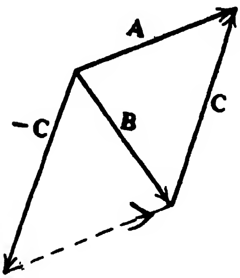


Fig. 7.

Rule.—*To form the vector difference of two vectors, reverse the direction of the subtrahend and then take the vector sum; or, if the minuend and subtrahend have the same origin, draw the vector connecting the two termini from the terminus of the subtrahend to the terminus of the minuend.*

12. Commutative and Associative Laws.—In a similar manner, it is very simple to prove that

$$\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A} \quad \text{(commutative law)}$$

and

$$\mathbf{A} + \mathbf{B} + \mathbf{C} = (\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C}) \quad \text{(associative law)},$$

which are equivalent to the statement that the order of compounding vectors is immaterial.

13. Multiplication of Vectors by Scalars.—It is evident that a vector can be multiplied by a scalar and that the result is again a vector. The multiplication alters the length of the vector, but not its direction. Thus, if a is a scalar, $a\mathbf{A}$ is a vector whose length is aA and whose direction is the same as that of **A**.

If m and n are two scalars, it is simple to prove that

$$m(n\mathbf{A}) = n(m\mathbf{A}) = (mn)\mathbf{A} \quad \text{(associative and commutative laws)},$$
$$(m + n)\mathbf{A} = m\mathbf{A} + n\mathbf{A} \quad \text{(distributive law of scalar factors)},$$
$$m(\mathbf{A} + \mathbf{B}) = m\mathbf{A} + m\mathbf{B} \quad \text{(distributive law of vectors)},$$
$$-(\mathbf{A}+\mathbf{B}) = -\mathbf{A} - \mathbf{B} \quad \text{(distributive law for negative sign)}.$$

Each of these statements should be worked out carefully by the student.

It follows, therefore, that *the laws which govern addition, subtraction, and scalar multiplication of vectors are identical with those governing these operations in ordinary algebra.* This fact permits the use of the notation and the operations of algebra for the corresponding operations with vectors.

14. Example.—Suppose there are given the vector equations

$$2\mathbf{A} + \mathbf{B} = \mathbf{M},$$
$$\mathbf{A} + 2\mathbf{B} = \mathbf{N},$$

nd it is desired to express **A** and **B** in terms of **M** and **N**. The quations can be solved just as though they were algebraic, with he result

$$\mathbf{A} = +\frac{2}{3}\mathbf{M} - \frac{1}{3}\mathbf{N},$$

$$\mathbf{B} = -\frac{1}{3}\mathbf{M} + \frac{2}{3}\mathbf{N},$$

which the student should verify geometrically.

Vectors that are parallel to the same straight line are said to be *collinear*. Two collinear vectors differ only by a scalar factor.

Vectors that are parallel to the same plane are said to be *coplanar*. Any *two* vectors are necessarily coplanar.

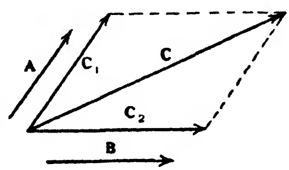


Fig. 8.

15. Three Coplanar Vectors.—If **A**, **B**, and **C** are coplanar vectors and **A** and **B** are not collinear, it is possible to express **C** in terms of **A** and **B**.

The vector **C** can be resolved into components $\mathbf{C}_1$ and $\mathbf{C}_2$ parallel to **A** and **B**, respectively, so that

$$\mathbf{C} = \mathbf{C}_1 + \mathbf{C}_2.$$

Since $\mathbf{C}_1$ and **A** are collinear,

$$\mathbf{C}_1 = a\mathbf{A},$$

where a is a scalar. Similarly,

$$\mathbf{C}_2 = b\mathbf{B},$$

and therefore

$$\mathbf{C} = a\mathbf{A} + b\mathbf{B}.$$

Hence, between any three coplanar vectors there exists a linear relationship of the form

$$a\mathbf{A} + b\mathbf{B} + c\mathbf{C} = 0,$$

where a, b, and c are scalars.

If $\mathbf{C} = 0$ and **A** and **B** are not collinear, then $a = 0$ and $b = 0$.

16. Vector Equation of a Straight Line.—If **A** and **B** are two given non-collinear vectors with the same origin O and the lines of these vectors are cut across by a third line l, the three sides of

the triangle so formed (Fig. 9) can be regarded as the thr vectors $a\mathbf{A}$, $b\mathbf{B}$, and $\mathbf{D}$; so that

$$\mathbf{D} = a\mathbf{A} - b\mathbf{B}.$$

Now let $\mathbf{C}$ be any vector whose origin is at O and whose terminus lies in l. Evidently,

$$\begin{aligned} \mathbf{C} &= b\mathbf{B} + t\mathbf{D}, \qquad \text{where} \qquad t = \frac{\overline{LM}}{\overline{LN}}, \\ &= b\mathbf{B} + t(a\mathbf{A} - b\mathbf{B}), \\ &= ta\mathbf{A} + (1 - t)b\mathbf{B}. \end{aligned}$$

If t is regarded as a variable parameter, $\mathbf{C}$ is a variable vector; but its terminus always lies in the line l. Hence,

$$\mathbf{C} = ta\mathbf{A} + (1 - t)b\mathbf{B} \tag{1}$$

is the vector equation of the straight line l.

The vector equation

$$\mathbf{C} = x\mathbf{A} + y\mathbf{B}$$

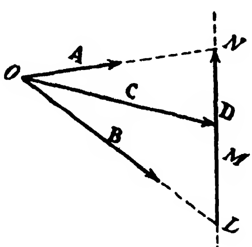


FIG. 9.

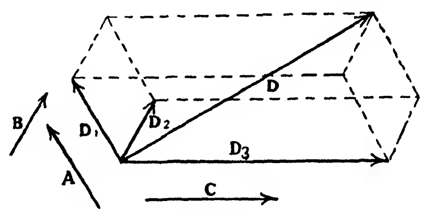


FIG. 10.

will be a straight line if x and y satisfy the algebraic equation

$$\frac{x}{a} + \frac{y}{b} = 1,$$

which is obtained by eliminating t between the equations

$$x = at, \qquad y = (1 - t)b.$$

If the line l is parallel to the vector $\mathbf{B}$, the parameter b is infinite. In this event $x = a$ and y is arbitrary, so that the equation of a line parallel to $\mathbf{B}$ is

$$\mathbf{C} = a\mathbf{A} + y\mathbf{B}.$$

17. Four Non-coplanar Vectors.—If $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ are non-coplanar, any vector in space $\mathbf{D}$ can be expressed in terms of $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$; for $\mathbf{D}$ can be resolved into three components $\mathbf{D}_1$, $\mathbf{D}_2$, and $\mathbf{D}_3$ parallel to $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ respectively, Fig. 10, so that

$$\mathbf{D} = \mathbf{D}_1 + \mathbf{D}_2 + \mathbf{D}_3.$$

Since $\mathbf{D}_1$ is collinear with $\mathbf{A}$, $\mathbf{D}_2$ with $\mathbf{B}$, $\mathbf{D}_3$ with $\mathbf{C}$, there exist three scalars a, b, and c which satisfy the relations

$$\mathbf{D}_1 = a\mathbf{A}, \qquad \mathbf{D}_2 = b\mathbf{B}, \qquad \mathbf{D}_3 = c\mathbf{C};$$

and therefore

$$\mathbf{D} = a\mathbf{A} + b\mathbf{B} + c\mathbf{C}.$$

Hence, between any four vectors in space there exists a linear relation of the general form

$$a\mathbf{A} + b\mathbf{B} + c\mathbf{C} + d\mathbf{D} = 0,$$

where a, b, c, and d are scalars.

18. Relations Independent of the Origin.—Suppose the system of vectors $\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{A}_3$, $\cdots$ has a common origin O with the termini at the points p_1, p_2, p_3, $\cdots$, and that between these vectors there exists the relation

$$a_1\mathbf{A}_1 + a_2\mathbf{A}_2 + a_3\mathbf{A}_3 + \cdots = 0. \quad (1)$$

Suppose further that this equation holds wherever the origin may be, provided the points p_1, p_2, p_3, $\cdots$ remain fixed. Under these conditions the vector equation (1) is said to be independent of the origin.

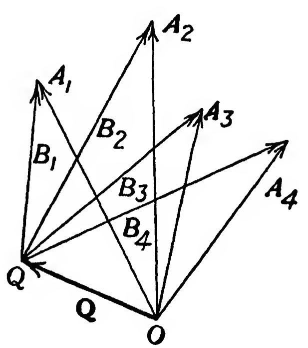


FIG. 11.

If the origin is changed from O to Q (Fig. 11) and if the vector $\overrightarrow{OQ}$ is denoted by $\mathbf{Q}$, $\overrightarrow{Qp_i}$ by $\mathbf{B}_i$, then

$$\mathbf{A}_1 = \mathbf{B}_1 + \mathbf{Q}, \qquad \mathbf{A}_2 = \mathbf{B}_2 + \mathbf{Q}, \qquad \mathbf{A}_3 = \mathbf{B}_3 + \mathbf{Q}, \qquad \cdots,$$

and Eq. (1) becomes

$$a_1(\mathbf{B}_1 + \mathbf{Q}) + a_2(\mathbf{B}_2 + \mathbf{Q}) + a_3(\mathbf{B}_3 + \mathbf{Q}) + \cdots = 0.$$

But, by hypothesis,

$$a_1\mathbf{B}_1 + a_2\mathbf{B}_2 + a_3\mathbf{B}_3 + \cdots = 0.$$

Therefore,

$$(a_1 + a_2 + a_3 + \cdots)\mathbf{Q} = 0,$$

from which it follows that

$$a_1 + a_2 + a_3 + \cdots = 0.$$

From this the following theorem is obtained: *A necessary and sufficient condition that the vector equation*

$$a_1\mathbf{A}_1 + a_2\mathbf{A}_2 + a_3\mathbf{A}_3 + \cdots = 0$$

shall be independent of the origin is that the algebraic equation

$$a_1 + a_2 + a_3 + \cdots = 0$$

shall hold.

It has been proved that this condition is necessary. It will be left to the student to prove that it is sufficient.

19. Examples.—It has already been seen that if the vectors **A**, **B**, and **C** have the same origin and the terminus of **C** lies in the straight line which passes through the termini of **A** and **B**,

$$\mathbf{C} = t\mathbf{A} + (1 - t)\mathbf{B},$$

or

$$t\mathbf{A} + (1 - t)\mathbf{B} - \mathbf{C} = 0.$$

For a given set of terminals it is evident from the manner in which this equation was derived that it is independent of the origin, and it will be observed that the sum of the coefficients is zero.

More generally, if **A**, **B**, and **C** are coplanar vectors with the same origin, and if

$$a\mathbf{A} + b\mathbf{B} + c\mathbf{C} = 0,$$

and

$$a + b + c = 0,$$

then **A**, **B**, and **C** terminate in the same straight line, for, since

$$b = -(a + c),$$

it follows that

$$a\mathbf{A} - (a + c)\mathbf{B} + c\mathbf{C} = 0,$$

or

$$a(\mathbf{A} - \mathbf{B}) = c(\mathbf{B} - \mathbf{C});$$

that is, the two vectors $(\mathbf{A} - \mathbf{B})$ and $(\mathbf{B} - \mathbf{C})$ are collinear, and since the terminus of **B** is common to both of them they lie in the same straight line.

The converse of this proposition also is true, that is, if **A**, **B**, and **C** have the same origin and terminate in the same straight line then there exist three numbers a, b, and c such that

$$a\mathbf{A} + b\mathbf{B} + c\mathbf{C} = 0,$$

and

$$a + b + c = 0.$$

A similar argument shows that if four vectors in space **A**, **B**, **C**, **D** have the same origin, a necessary and sufficient condition that their termini lie in the same plane is that

$$a\mathbf{A} + b\mathbf{B} + c\mathbf{C} + d\mathbf{D} = 0,$$

with

$$a + b + c + d = 0.$$

20. Centroids.—Let there be given in space a set of points p_i, $i = 1, \cdots, n$. From any point O draw the vectors $\mathbf{A}_i = \overrightarrow{Op_i}$. Let the vector $\mathbf{C}$ be defined by the relation

$$\mathbf{C} = \frac{1}{n}(\mathbf{A}_1 + \mathbf{A}_2 + \cdots + \mathbf{A}_n). \tag{1}$$

The terminus of the vector $\mathbf{C}$ defines a new point p. If all of the terms of Eq. (1) are taken to the same side of the equality sign, then the sum of the coefficients of all of the vectors is zero. Hence the point p, thus defined, is independent of the choice of the origin. It is called the *centroid of the set of points* p_i.

21. Rectangular Coordinates of the Centroid.—Let $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ be three vectors each of length unity and each perpendicular to the other two. Each of the vectors $\mathbf{A}_s$ can be resolved into its components in these three directions. Thus

$$\mathbf{A}_s = x_s\mathbf{i} + y_s\mathbf{j} + z_s\mathbf{k}, \qquad s = 1, 2, \cdots, n, \tag{1}$$

where the scalars x_s, y_s, z_s are nothing other than the rectangular coordinates of the point p_s. On adding together all of the equations of Eq. (1) and then dividing by n, it is found that

$$\mathbf{C} = \left(\frac{x_1 + x_2 + \cdots + x_n}{n}\right)\mathbf{i} + \left(\frac{y_1 + y_2 + \cdots + y_n}{n}\right)\mathbf{j} + \left(\frac{z_1 + z_2 + \cdots + z_n}{n}\right)\mathbf{k},$$

which says that the rectangular coordinates of the centroid p, *viz.*,

$$\bar{x} = \frac{x_1 + x_2 + \cdots + x_n}{n}, \qquad \bar{y} = \frac{y_1 + y_2 + \cdots + y_n}{n},$$
$$\bar{z} = \frac{z_1 + z_2 + \cdots + z_n}{n},$$

are the averages of the corresponding coordinates of the points p_i, whatever the orientation of the vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ may be.

22. The Weighted Centroid.—Suppose each of the points p_i is to be counted m_i times. The vector analogous to Eq. (20.1)* is

$$\mathbf{C} = \frac{m_1\mathbf{A}_1 + m_2\mathbf{A}_2 + \cdots + m_n\mathbf{A}_n}{m_1 + m_2 + \cdots + m_n}.$$

* In references to previous equations the integral part of the reference number is the section number and the decimal part is the equation number. The section number is not given when the reference is to the current section.

The numbers m_i are called the *weighting factors*, and the terminus of the vector **C**, thus defined, is the centroid of the set of points p_i for the weighting factors m_i. It is evidently independent of the choice of the origin.

In terms of the rectangular components, it is found, just as before, that

$$\mathbf{C} = \left(\frac{m_1x_1 + m_2x_2 + \cdots + m_nx_n}{m_1 + m_2 + \cdots + m_n}\right)\mathbf{i} +$$
$$\left(\frac{m_1y_1+m_2y_2+ \cdots +m_ny_n}{m_1 + m_2 + \cdots + m_n}\right)\mathbf{j} + \left(\frac{m_1z_1+m_2z_2+ \cdots m_nz_n}{m_1 + m_2 + \cdots + m_n}\right)\mathbf{k},$$

which gives the rectangular coordinates of the weighted centroid.

23. Application of Vector Methods to Geometry.—For the purpose of obtaining familiarity with vectors and gaining facility in their use, it will be helpful to apply vector methods to the solution of problems in elementary geometry, a subject with which, it is assumed, the student is already familiar.

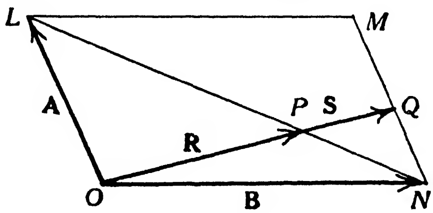


FIG. 12.

Example 1.—The line which joins one corner of a parallelogram to the middle point of an opposite side trisects the diagonal and is trisected by it.

Let the parallelogram be $LMNO$, and let the point Q bisect the side MN. Let the lines LN and OQ intersect in the point P. It is desired to show that $PN/LN = PQ/OQ = 1/3$.

Let $\overrightarrow{OL} = \mathbf{A}$, $\overrightarrow{ON} = \mathbf{B}$, $\overrightarrow{OP} = \mathbf{R}$, and $\overrightarrow{OQ} = \mathbf{S}$. Since **R** and **S** are collinear

$$\mathbf{R} = x\mathbf{S},$$

where $x = \overline{OP}/\overline{OQ}$ is some unknown number.

Since the terminus of **R** lies in the diagonal LN,

$$\mathbf{R} = t\mathbf{A} + (1 - t)\mathbf{B}, \qquad \text{where} \qquad t = \frac{\overline{PN}}{\overline{LN}};$$

also

$$\mathbf{S} = \frac{1}{2}\mathbf{A} + \mathbf{B}.$$

On multiplying the second of these equations by x and subtracting from the first, there results

$$\mathbf{R} - x\mathbf{S} = 0 = \left(t - \frac{1}{2}x\right)\mathbf{A} + (1 - t - x)\mathbf{B}.$$

Since **A** and **B** are not collinear,

$$t - \frac{1}{2}x = 0, \qquad 1 - t - x = 0.$$

Therefore,

$$t = \frac{1}{3}, \qquad x = \frac{2}{3}.$$

Hence,

$$\frac{\overline{PN}}{\overline{LN}} = \frac{\overline{PQ}}{\overline{OQ}} = \frac{1}{3},$$

and the lines LN and OQ trisect each other.

Example 2.—If through any point within a triangle lines be drawn parallel to the three sides, terminating at the sides, the sum of the ratios of these lines to their corresponding sides is 2.

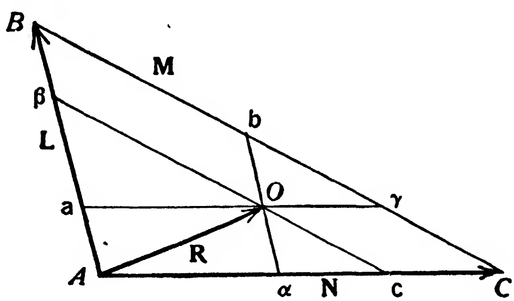


FIG. 13.

Let ABC be the triangle (Fig. 13). Through the interior point O, draw

$a\gamma$ parallel to AC,
$b\alpha$ parallel to BA,
and $c\beta$ parallel to CB.

It is desired to prove that

$$\frac{a\gamma}{AC} + \frac{b\alpha}{BA} + \frac{c\beta}{CB} = 2.$$

Let $\overrightarrow{AB} = \mathbf{L}$, $\overrightarrow{BC} = \mathbf{M}$, $\overrightarrow{AC} = \mathbf{N}$, $\overrightarrow{AO} = \mathbf{R}$,

$$\frac{\overline{Aa}}{\overline{AB}} = m, \qquad \frac{\overline{A\alpha}}{\overline{AC}} = n.$$

Then

$$\frac{a\gamma}{AC} = \frac{aB}{AB} = 1 - m, \qquad \frac{b\alpha}{BA} = \frac{\alpha C}{AC} = 1 - n.$$

Now

$$\begin{aligned} \mathbf{R} &= m\mathbf{L} + n\mathbf{N} \\ &= (m + n)\mathbf{L} + n(\mathbf{N} - \mathbf{L}) \\ &= (m + n)\mathbf{L} + n\mathbf{M}. \end{aligned}$$

But, taken directly from the figure,

$$\begin{aligned} \mathbf{R} = \overrightarrow{A\beta} + \overrightarrow{\beta O} &= \frac{A\beta}{AB}\mathbf{L} + \frac{Bb}{BC}\mathbf{M} \\ &= \frac{c\beta}{CB}\mathbf{L} + \frac{A\alpha}{AC}\mathbf{M}. \end{aligned}$$

On comparing these two expressions for **R**, it is seen that

$$\frac{c\beta}{CB} = m + n,$$

and, therefore,

$$\frac{a\gamma}{AC} + \frac{b\alpha}{BA} + \frac{c\beta}{CB} = (1 - m) + (1 - n) + (m + n) = 2.$$

Example 3.—If through any point within a parallelogram two lines be drawn parallel to the sides, the diagonals of the smaller parallelograms thus formed intersect upon the diagonal of the given parallelogram.

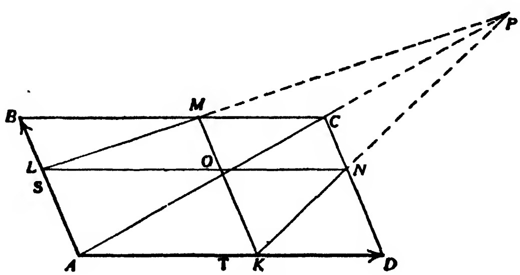


FIG. 14.

Let $ABCD$ be the parallelogram and let LN and KM be drawn parallel to BC and CD, respectively. It is desired to prove that the diagonals LM and KN intersect at a certain point P on the diagonal AC.

For this purpose, let

$$\mathbf{S} = \overrightarrow{AB}, \qquad \mathbf{T} = \overrightarrow{AD}, \qquad m = \frac{AL}{AB}, \qquad n = \frac{AK}{AD}.$$

Since the intercepts of line LM on the lines of the vectors **S** and **T** are $m\mathbf{S}$ and $-\frac{mn}{1-m}\mathbf{T}$, respectively, any vector, **V**, whose origin is at A and whose terminus lies in the line LM can be written (by Sec. 16)

$$\mathbf{V} = xm\mathbf{S} - (1-x)\frac{mn}{1-m}\mathbf{T}.$$

Likewise, the intercepts of the line KN are $-mn/(1-n)\mathbf{S}$ and $n\mathbf{T}$. Hence, the equation of any vector **W** whose origin is at A and whose terminus is in the line KN, is

$$\mathbf{W} = -y\frac{mn}{1-n}\mathbf{S} + (1-y)n\mathbf{T}.$$

At the intersection of the lines LM and KN the vectors **V** and **W** are identical and, therefore, the corresponding coefficients in the two expressions are equal. That is,

$$mx = -\frac{mn}{1-n}y, \qquad (1-x)\frac{mn}{1-m} = -(1-y)n.$$

The solution of these equations is

$$x = \frac{n}{m+n-1}, \qquad y = \frac{n-1}{m+n-1}.$$

These values substituted in the expression for either **V** or **W** give for the vector $\overrightarrow{AP}$

$$\overrightarrow{AP} = \frac{mn}{m+n-1}(\mathbf{S}+\mathbf{T}),$$

which shows that $\overrightarrow{AP}$ is collinear with $\overrightarrow{AC} = \mathbf{S} + \mathbf{T}$; and since they have the same origin A, they lie in the same straight line.

In a similar manner it is proved that the diagonals KL and MN intersect on the diagonal BD.

Problems I

1. The vectors **A** and **B** have a common origin. **A** is one component of a rectangular resolution of the vector **P**, and so also is **B**. Show how to construct the vector **P**.

2. Three vectors, **A**, **B**, and **C**, arranged so that the terminus of each is the origin of the next following, form a closed triangle. Show that

$$\mathbf{A} + \mathbf{B} + \mathbf{C} = 0.$$

Generalize to n vectors.

3. Prove the converse of problem 2.

4. Show that the vector equation of a line through the origin has the form

$$\mathbf{C} = t(a\mathbf{A} + b\mathbf{B})$$

and interpret the parameters a, b, and t.

5. Using vector methods, prove that the diagonals of a parallelogram bisect each other.

6. Prove that the medians of a triangle considered as vectors can be arranged in the form of a closed triangle.

7. Prove that the medians of a triangle meet in a point which trisects each of them.

8. Prove that the components of a sum of vectors are equal to the sum of the components of the individual vectors.

9. If A and B are the lengths of the vectors **A** and **B**, respectively, and if **A**, **B**, and **C** have the same origin, then the vector

$$\mathbf{C} = B\mathbf{A} + A\mathbf{B}$$

bisects the angle between **A** and **B**.

10. The bisectors of the angles of a triangle meet in a point.

11. If **A**, **B**, and **C** form a closed triangle and **R** has the same origin as **A** and **B**, then

$$\mathbf{R} = \frac{B}{A+B}\mathbf{A} + \frac{A}{A+B}\mathbf{B}$$

bisects the angle between **A** and **B** and terminates in **C**.

12. The sides of a right triangle are α, β, γ with the right angle opposite α. The perpendicular from the right angle to α is δ. If **b**, **c**, and **d** are unit vectors along the lines β, γ, δ, show that

$$\frac{\mathbf{b}}{\beta} + \frac{\mathbf{c}}{\gamma} = \frac{\mathbf{d}}{\delta}.$$

13. If **A**, **B**, **C** form a closed triangle, the vector **P** which has the same origin as **A** and **B**, is perpendicular to **C** and terminates in **C**, is

$$\mathbf{P} = \frac{B^2 + C^2 - A^2}{2C^2}\mathbf{A} + \frac{A^2 + C^2 - B^2}{2C^2}\mathbf{B}.$$

14. Prove that the perpendiculars from the three vertices of a triangle to the opposite sides meet in a point.

15. If, in problem 13, the vector **P** terminates at the intersection of the three perpendiculars (the orthocenter) its expression is

$$\mathbf{P} = (B^2 + A^2 - C^2)\frac{(B^2 + C^2 - A^2)\mathbf{A} + (A^2 + C^2 - B^2)\mathbf{B}}{2A^2B^2 + 2B^2C^2 + 2C^2A^2 - A^4 - B^4 - C^4}.$$

16. The vector to the center of the circumscribing circle terminating there is

$$\mathbf{R} = \frac{B^2(A^2 + C^2 - B^2)\mathbf{A} + A^2(B^2 + C^2 - A^2)\mathbf{B}}{2A^2B^2 + 2B^2C^2 + 2C^2A^2 - A^4 - B^4 - C^4}.$$

17. If the angle between two vectors **A** and **B** is 2φ, and if the angle θ is measured from the bisector of this angle, then any vector **C** which makes

an angle θ with this bisector is represented in terms of **A** and **B** by the formula

$$\frac{\mathbf{C}}{C} = \frac{\sin(\varphi + \theta)}{\sin 2\varphi}\frac{\mathbf{A}}{A} + \frac{\sin(\varphi - \theta)}{\sin 2\varphi}\frac{\mathbf{B}}{B}.$$

18. The intersection of the three medians of a triangle is the centroid of the three vertices.

19. The origin O of a system of vectors $\mathbf{F}_i$ is at the centroid of their termini. The lines l_i in which the vectors lie are cut by any transversal at distances L_i from O. A distance L_k is to be taken negatively if the transversal cuts l_k on the opposite side of O from $\mathbf{F}_k$. Show that

$$\sum \frac{F_i}{L_i} = 0.$$

20. If A, B, and C are the vertices of a triangle with sides a, b, and c, and I is the center of the inscribed circle, then I is also the centroid of the points A, B, C for the multipliers a, b, and c, respectively.

CHAPTER II

VELOCITY

24. Motion.—The concept of motion involves both the notion of speed and the notion of direction. The simplest type of motion is uniform motion along a straight line. The *average speed* of a particle moving in a straight line is defined to be the quotient of the distance covered by the time required in covering it;

$$\text{average speed} = \frac{\text{distance}}{\text{time}}.$$

If this quotient has the same value for every segment of the straight line, irrespective of the length of the segment, the speed is constant. The motion, therefore, can be represented by a directed straight line, the direction of which is parallel to the line of motion and the length of which is equal to the speed. This directed quantity is the *velocity* of the particle.

25. Composition of Constant Velocities.—A particle may have two or more such velocities at the same time. Consider, for example, a small bug crawling along a sheet of paper with a constant velocity $\mathbf{V}_1$. Suppose the sheet of paper also, is being translated (but not rotated) with a constant velocity $\mathbf{V}_2$, with respect to the table on which it lies. With respect to the table the bug has two velocities; for, evidently, he partakes of the motion of the paper.

Consider a short interval of time dt. During this interval the bug has two displacements $\mathbf{D}_1 = \mathbf{V}_1 dt$ and $\mathbf{D}_2 = \mathbf{V}_2 dt$. These two displacements are equivalent to a single displacement (Fig. 15)

$$\mathbf{D}_3 = \mathbf{D}_1 + \mathbf{D}_2$$

of the bug with respect to the table. If the velocity $\mathbf{V}_3$ is defined by the relation $\mathbf{D}_3 = \mathbf{V}_3 dt$, it is evident also that

$$\mathbf{V}_3 = \mathbf{V}_1 + \mathbf{V}_2,$$

and $\mathbf{V}_3$ is the velocity of the bug with respect to the table. Since $\mathbf{V}_1$ and $\mathbf{V}_2$ are constant it follows that $\mathbf{V}_3$ also is constant. The

parallelogram of velocities differs from the parallelogram of displacements only in scale, for the velocities and the displacements differ only by the scalar factor dt. Since displacements are vectors (Sec. 8), it is evident that constant velocities also are vectors.

26. Variable Speeds.—It may happen for straight-line motion that the quotient, distance divided by time, depends upon the particular segment that is chosen for measurement. In this event, the speed is certainly not constant, and the quotient merely gives the average speed over the segment measured.

Imagine the point P at the center of a segment AB (Fig. 16) of the line. The shorter this segment the more nearly will the

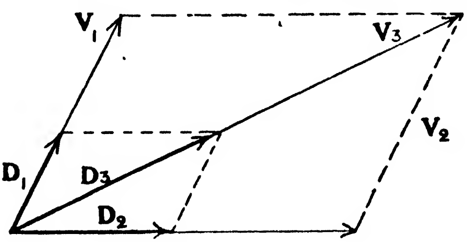


FIG. 15.

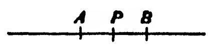


FIG. 16.

average speeds over subsegments, which contain P as a center point, agree. The *speed of the particle at the point* P is defined as the limit of the average speed over these subsegments as the lengths of the subsegments tend to zero. If the particle is at the point P at the particular instant t_p, this limit is also called *the speed of the particle at the instant* t_p. The velocity of the particle is variable, for even though the direction of the motion is constant the speed varies from one point P to another.

In the notation of the calculus, if x is the coordinate of P and x_1 and x_2 are the end points of the subsegments which contain P, then the speed at P is

$$\text{Speed} = \text{limit}\,\frac{x_2 - x_1}{t_2 - t_1} = \frac{dx}{dt}. \tag{1}$$

27. Curvilinear Motion.—If a particle is moving along a curved path its speed at a point O is defined in a manner analogous to the speed at a point in a straight line.

Suppose the particle is moving along the curve $OA_1A_2A_3A_4$ and that it arrives at the points A_1, A_2, A_3, A_4 after passing the point O at the expiration of the intervals of time t_1, t_2, t_3, t_4, respec-

tively. The average velocity during the interval of time t_4 is the displacement vector $\overline{OA}_4$ divided by the time t_4, namely, the vector $\overline{OB}_4$; for if the particle had had the velocity $\overline{OB}_4$ when it was at the point O, and had maintained that velocity constantly, it would have arrived at the point A_4 after the interval of time t_4.

Similar statements hold for the points A_3, A_2, and A_1. Since $t_3 < t_4$, it is evident that $\overline{OB}_3$ compared with $\overline{OA}_3$ is longer than $\overline{OB}_4$ compared with $\overline{OA}_4$. The points B_4, B_3, $\cdots$ lie upon some curve L. As the time intervals tend toward zero, the average velocity vectors tend toward tangency to the curve OA_4, while the terminus of the average velocity vector moves toward the point B_0 where the tangent at O intersects the curve L. This limiting value of the average velocity vector is, by definition, the velocity of the particle at the point O.

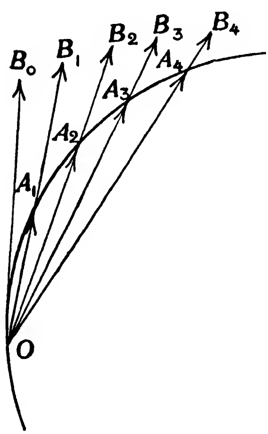


FIG. 17.

28. Definition.—*Velocity is the rate of change of position, and is a directed quantity.*

29. Composition of Variable Velocities.—A particle may have two variable velocities. Suppose a small bug is crawling on a sheet of paper with variable speed along a curved line. At the same time the sheet of paper is being translated along the surface of the table with variable speed along a curved line on the table. The bug partakes of both motions.

To analyse the motion with respect to the table, consider the positions of the bug with respect to the table at the instants t and $t + dt$. Let it be supposed that the interval of time dt is very small. During this interval the bug will have undergone two small displacements $\mathbf{D}_1$ and $\mathbf{D}_2$, which together are equivalent to the single displacement $\mathbf{D}_3 = \mathbf{D}_1 + \mathbf{D}_2$. If these small displacements are divided by the small interval of time dt, the average velocities obtained are $\mathbf{V}_1$, $\mathbf{V}_2$, and $\mathbf{V}_3$; that is, velocities which, had they been constant, would have produced the displacements $\mathbf{D}_1$, $\mathbf{D}_2$, and $\mathbf{D}_3$ in the interval of time dt. Evidently

$$\mathbf{V}_3 = \mathbf{V}_1 + \mathbf{V}_2.$$

If a sequence of such small intervals of time, dt, which have the limit zero, and the corresponding set of diagrams are imagined, it will be seen that the displacements $\mathbf{D}_1$, $\mathbf{D}_2$, and $\mathbf{D}_3$ not only tend toward zero, but also tend toward definite directions, namely, the directions of the tangents to the two curves mentioned at the instant t. The average velocity vectors $\mathbf{V}_1$, $\mathbf{V}_2$, and $\mathbf{V}_3$ tend toward definite lengths and directions; $\mathbf{V}_1$ and $\mathbf{V}_2$ tend toward the velocities of the bug and the paper, respectively, at the instant t, and $\mathbf{V}_3$ tends toward the velocity of the bug with respect to the table. But at all stages of the sequence, $\mathbf{V}_3 = \mathbf{V}_1 + \mathbf{V}_2$. That is, at the limit the velocity of the bug with respect to the table is the vector sum of the velocity of the bug with respect to the paper and the velocity of the paper with respect to the table.

It is evident therefore that *velocities are true vectors.* They compound according to the parallelogram law.

Rule I.—*If the velocity of B with respect to A is* $\mathbf{V}_1$ *and the velocity of C with respect to B is* $\mathbf{V}_2$, *then the velocity of C with respect to A is*

$$\mathbf{V}_3 = \mathbf{V}_1 + \mathbf{V}_2.$$

Rule II.—*If the velocity of B with respect to A is* $\mathbf{V}_1$, *and the velocity of C with respect to A is* $\mathbf{V}_2$, *then the velocity of C with respect to B is*

$$\mathbf{V}_3 = \mathbf{V}_2 - \mathbf{V}_1.$$

30. Units.—Many systems of units are in common use for expressing speeds. The speed of a runner may be expressed in yards per minute; the speed of a railroad train in miles per hour; the speed of a rifle bullet in feet per second, etc. In the scientific or c.g.s (centimeter-gram-second) system, it is expressed in centimeters per second.

It goes without saying that in compounding velocities, all velocities must be expressed in the same units.

Example 1 (*Numerical*).—The steamer *Selwyn* is making 18 knots on a course 10° north of east. The steamer *Bernice* is making 14 knots on a course 30° east of south. What is the velocity of the *Selwyn* with respect to the *Bernice?*

Let $\mathbf{S}$ be the velocity of the *Selwyn*, and S its speed,

$\mathbf{B}$ be the velocity of the *Bernice*, and B *its speed,*

and $\mathbf{V}$ the velocity of the *Selwyn* with respect to the *Bernice.* Then, by rule II,

$$\mathbf{V} = \mathbf{S} - \mathbf{B}.$$

The lengths of the sides of the closed triangle (Fig. 18) are

$$S = 18, \qquad B = 14, \qquad V = ?$$

The angle between **S** and **B**, $(\widehat{\mathbf{SB}})$, is 70°. Consequently

$$V^2 = S^2 + B^2 - 2SB \cos 70°,$$
$$= 324 + 196 - 2 \cdot 18 \cdot 14 \cdot 0.3420 = 347.6,$$

and $\quad V = 18.64$ knots.

Also,

$$\frac{18}{\sin \sigma} = \frac{14}{\sin \beta} = \frac{18.66}{\sin 70°}.$$

Hence,

$$\sin \beta = \frac{14 \cdot \sin 70°}{18.66},$$

and

$$\beta = 44° \ 53.'5, \qquad \sigma = 65° \ 6.'5, \qquad \alpha = 35° \ 6.'5$$

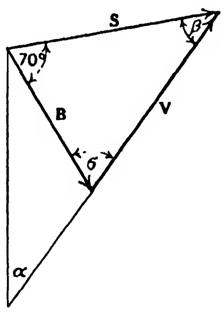


Fig. 18.

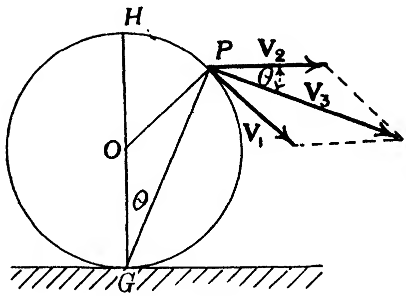


Fig. 19.

With respect to the *Bernice*, the course of the *Selwyn* is north, 35°10.'5 east, and its speed is 18.66 knots.

Example 2.—What is the velocity of a point on the rim of a rolling wheel if the center of the wheel is moving forward with a speed V?

With respect to its axle the wheel is merely turning around, and all points of the rim have the same speed in this motion, although different points move in different directions. The *speeds* are the same but the *velocities* are different.

Let G be the point in contact with the ground, and therefore at rest with respect to the ground. Its speed with respect to the axle is V, since the speed of the ground with respect to the axle is V. Hence, in the rotation about the axle, all points of the rim have the speed V. In the diagram the wheel is supposed to be rolling toward the right, and therefore the wheel is turning

clockwise with respect to the axle. A point P on the rim has a velocity $\mathbf{V}_1$ with respect to the axle, the length of $\mathbf{V}_1$ being V and its direction is tangent to the rim. The axle, however, has a velocity $\mathbf{V}_2$ with respect to the ground. By rule I, therefore, the velocity of P with respect to the ground is

$$\mathbf{V}_3 = \mathbf{V}_1 + \mathbf{V}_2,$$

the magnitude of $\mathbf{V}_2$ being the same as the magnitude of $\mathbf{V}_1$, namely V.

The vector $\mathbf{V}_1$ is perpendicular to the radius OP, and $\mathbf{V}_2$ is perpendicular to OH. Since $\mathbf{V}_3$ bisects the angle between $\mathbf{V}_1$ and $\mathbf{V}_2$, it is perpendicular to the bisector of the angle POH and therefore perpendicular to PG. That is, the point P is moving just as though it were pivoting on the point G.

If θ is the angle which PG makes with the vertical, it is easily verified that

$$V_3 = 2V \cos \theta.$$

The highest point of the wheel, for which $\theta = 0$, has the greatest speed with respect to the ground; in fact, just twice the speed of the axle with respect to the ground.

Example 3.—Three horses in a field are at the vertices of an equilateral triangle. Their motions relative to a person riding along a road with a speed V are in the directions of the sides of the triangle (in the same sense) and with the same speed V. Show that the three horses are moving with respect to the ground along concurrent lines.

Since the velocities of the horses are given relative to the man, and the velocity of the man relative to the ground, rule I applies.

Let $\mathbf{V}$ be the velocity of the man relative to the ground and let $\mathbf{L}$, $\mathbf{M}$, and $\mathbf{N}$ be the velocities of the horses located at A, B, and C, respectively, relative to the man (Fig. 20), with

$$L = M = N = V.$$

Then the velocities of the horses relative to the ground are

$$\mathbf{V}_1 = \mathbf{L} + \mathbf{V}, \qquad \mathbf{V}_2 = \mathbf{M} + \mathbf{V}, \qquad \mathbf{V}_3 = \mathbf{N} + \mathbf{V}.$$

It will be assumed that the length of the side of the triangle is s, and it will be proved that the lines of the vectors $\mathbf{V}_1$, $\mathbf{V}_2$, and $\mathbf{V}_3$ meet at a point P.

First Solution.—At A draw a vector $\mathbf{K} = -\mathbf{N}$, and let $\mathbf{K}$ and $\mathbf{L}$ be the fundamental vectors in terms of which the other vectors are to be expressed. Let the angle beween $\mathbf{V}$ and $\mathbf{K}$ be 2α. Let F be the position of the man. At F draw lines parallel

to **L** and **K**, and at G draw GH parallel to **L**. Let $\overrightarrow{FG} = \mathbf{V}$. In the triangle FGH

$$\frac{FH}{\sin(60^\circ - 2\alpha)} = \frac{HG}{\sin 2\alpha} = \frac{V}{\sin 120^\circ}.$$

Hence,

$$\overrightarrow{FH} = \frac{\sin(60^\circ - 2\alpha)}{\sin 120^\circ}\mathbf{K}, \qquad \overrightarrow{HG} = \frac{\sin 2\alpha}{\sin 120^\circ}\mathbf{L}.$$

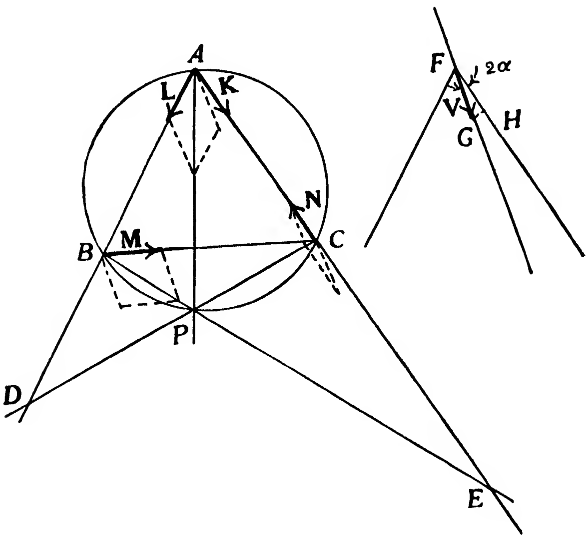


FIG. 20.

Since **V** is the sum of these two vectors and $\mathbf{V}_1 = \mathbf{V} + \mathbf{L}$,

$$\mathbf{V} = \frac{2\sin 2\alpha}{\sqrt{3}}\mathbf{L} + \frac{2\sin(60^\circ - 2\alpha)}{\sqrt{3}}\mathbf{K},$$

$$\mathbf{V}_1 = \left(1 + \frac{2\sin 2\alpha}{\sqrt{3}}\right)\mathbf{L} + \frac{2\sin(60^\circ - 2\alpha)}{\sqrt{3}}\mathbf{K},$$

or

$$\mathbf{V}_1 = \frac{4}{\sqrt{3}}\cos(30^\circ - \alpha)[\sin(30^\circ + \alpha)\mathbf{L} + \sin(30^\circ - \alpha)\mathbf{K}].$$

It is readily verified that $\widehat{\mathbf{V}_3\mathbf{N}} = 90^\circ - \alpha$. Let D be the point where the line of $\mathbf{V}_3$ intersects AB extended. Then from the triangle ACD, it is found that

$$\frac{AD}{\sin(90^\circ - \alpha)} = \frac{s}{\sin(30^\circ + \alpha)},$$

and therefore $$AD = \frac{s \cos \alpha}{\sin (30^\circ + \alpha)}.$$

Consequently the general expression (Eq. (16.1)) for the vector whose origin is at A and whose terminus lies in the line CD is

$$\mathbf{R} = \frac{ts \cos \alpha}{v \sin (30^\circ + \alpha)}\mathbf{L} + (1 - t)\frac{s}{v}\mathbf{K}.$$

Similarly, since $\mathbf{V}_2$ makes an angle of $90^\circ + \alpha$ with the side BA, and E is the point where the line of $\mathbf{V}_2$ intersects the side AC extended,

$$AE = \frac{s \cos \alpha}{\sin (30^\circ - \alpha)}.$$

The general expression for the vector whose origin is at A and whose terminus lies in the line BE is

$$\mathbf{T} = x\frac{s}{v}\mathbf{L} + (1 - x)\frac{s \cos \alpha}{v \sin (30^\circ - \alpha)}\mathbf{K}.$$

At the intersection P of the two lines CD and BE, the vectors $\mathbf{R}$ and $\mathbf{T}$ are identical. Hence, on comparing coefficients,

$$x\frac{s}{v} = \frac{ts \cos \alpha}{v \sin (30^\circ + \alpha)}, \qquad (1 - t)\frac{s}{v} = (1 - x)\frac{s \cos \alpha}{v \sin (30^\circ - \alpha)};$$

and therefore

$$t = \frac{4}{3}\sin^2 (30^\circ + \alpha), \qquad x = \frac{4}{3}\cos \alpha \sin (30^\circ + \alpha).$$

On substituting these values in the expression for either $\mathbf{R}$ or $\mathbf{T}$, it is found that

$$\begin{aligned}\overrightarrow{AP} &= \frac{4}{3}\frac{s}{v}\cos \alpha[\sin (30^\circ + \alpha)\mathbf{L} + \sin (30^\circ - \alpha)\mathbf{K}],\\ &= \frac{1}{\sqrt{3}}\frac{s}{v}\frac{\cos \alpha}{\cos (30^\circ - \alpha)}\mathbf{V}_1.\end{aligned}$$

It is therefore collinear with $\mathbf{V}_1$ and, since it has the same origin A, the two vectors lie in the same straight line. The line of $\mathbf{V}_1$, therefore, also passes through P and the lines of the three vectors $\mathbf{V}_1$, $\mathbf{V}_2$, and $\mathbf{V}_3$ are concurrent.

Second Solution.—The angle BCP equals $30^\circ - \alpha$, and the angle CBP equals $30^\circ + \alpha$. Therefore, the angle $CPB = 120^\circ$. Regarding the direction of $\mathbf{V}$ as a variable, the angle α is a variable, and so also is the position of the point P. The points B and C, however, remain fixed, and the angle $CPB = 120^\circ$ remains constant. Hence, the point P lies on an arc of a circle which passes through B and C, and since the angle at P is always 120°

the circle also passes through the point A. Hence, the lines of $\mathbf{V}_2$ and $\mathbf{V}_3$ intersect on the circle which circumscribes the triangle ABC.

Likewise, the lines of $\mathbf{V}_1$ and $\mathbf{V}_2$ intersect on the circumscribing circle, and so also do the lines of $\mathbf{V}_1$ and $\mathbf{V}_3$. Aside from the points A, B, and C these lines cut the circle but once each. Hence, they must all cut in the same point, and the lines therefore are concurrent.

Problems II

1. A certain distance is covered by a man walking 4 miles per hour in 18 min. less than by a man walking 2.5 miles per hour. What is the distance? *Ans.* 2 miles.

2. A passenger on a railroad train observes that a train 528 ft. long on a parallel track requires 6 sec. to pass him. If the two trains are traveling in opposite directions with equal speed, what is the speed of each train? *Ans.* 30 miles per hour.

3. A particle moves with constant speed around the circumference of a circle in the same time that a second particle moves uniformly across a diameter. Compare their speeds. *Ans.* As π is to 1.

4. What is the speed of a particle on the surface of the earth in latitude l due to the rotation of the earth, assuming the radius of the earth to be 3960 miles and the length of the sidereal day to be 86,164 sec.? *Ans.* $1524.7 \cos l$ ft. per second.

5. A man standing on the top of a train, which is moving at the rate of 30 miles per hour, throws a ball with a speed of 22 ft. per second in a direction perpendicular to the train. What is the speed and direction of the ball with respect to the track? *Ans.* Speed $= 22 \times \sqrt{5}$ ft. per second; angle $= \tan^{-1} 1/2$.

6. When a steamer is in motion it is found that an awning 8 ft. above the deck protects from rain the portion of the deck which is more than 4 ft. behind the vertical projection of the edge of the awning; but when the steamer comes to rest the line of separation of the wet and dry parts is 6 ft. in front of this projection. If the speed of the rain is 20 ft. per second, what is the speed of the steamer? *Ans.* 20 ft. per second.

7. If a wheel is rolling along a horizontal road, is there any point of the rim which has a velocity which is straight up or straight down? *Ans.* No.

8. In what direction must a boat be steered across a river which flows at the rate of 3 miles per hour by a man who rows 4 miles per hour in order to make a course at right angles to the bank? *Ans.* At $\sin^{-1} 3/4$ upstream.

9. Two particles move uniformly in straight lines. At a given time the distance between them is $\mathbf{D}$ and their relative velocity is $\mathbf{V}$, the components of which in the direction of $\mathbf{D}$ and perpendicular to it are $\mathbf{V}_1$ and $\mathbf{V}_2$. Show that when they are nearest together their distance is DV_2/V, and that they arrive in this position after the interval of time DV_1/V^2.

10. A steamer sailing by compass has a velocity of 20 knots due south in a current making 3 knots to the east. The wind is blowing 15 miles per hour from the southwest. Assuming that the smoke does not partake of the motion of the steamer, what is the velocity of the smoke with respect to the steamer? (A knot is 1 nautical mile = 6080 ft. per hour.) *Ans.* 50.4 ft. per second in a direction N. 12° 17′ E.

11. A balloon which is rising with a speed of $\sqrt{21}$ ft. per second is carried northward by the wind with a speed of 8 ft. per second. What is the velocity of the balloon with respect to a person who is walking westward with a speed of 6 ft. per second? *Ans.* Speed = 11 ft. per second; direction, altitude 24° 37′, azimuth N. 36° 52′ E.

12. If an aeroplane flies at the rate of 100 miles per hour in still air, how long will it take to fly around a square each of whose sides is 6 miles, (*a*) when the wind is not blowing, (*b*) when it is blowing 28 miles per hour parallel to two of the sides of the square, (*c*) when it is blowing 28 miles per hour parallel to a diagonal? *Ans.* (*a*) 14.4 min.; (*b*) 15 5/16 min.; (*c*) 15.31 min.

13. If a bicyclist rides faster than the wind, show that the wind will always seem to be against him in whatever direction he may ride.

CHAPTER III

ACCELERATION

31. Definition.—*Acceleration is the rate of change of velocity.* In ordinary language, an object is said to be *accelerated* if its speed is increased; it is *retarded* if its speed is diminished. In mechanics the single term *acceleration* takes into account not only the change in speed but also the change in the direction of motion.

It will be observed that it is not the rate of change of speed. It is the rate of change of velocity. If a particle is moving in a straight line, its velocity changes only as its speed changes, the direction of the motion being constant. In this case the rate of change of speed, and the rate of change of velocity are identical. If, however, a particle is moving uniformly in a circle, its speed is constant, and therefore its rate of change of speed is zero; but its velocity is always changing since the direction of the motion is always changing, and therefore its rate of change of velocity is never zero.

In mechanics *acceleration* always refers to the change in the vectors and not, as in common conversation, to the change in the scalars.

32. The Hodograph.—Consider a particle which is moving with constant or variable speed along a curve, and imagine a vector with a fixed point O as origin to represent the velocity of the particle. As the particle moves along its curve C, the velocity vector changes its length and direction, so that its terminus decribes a curve H, which is called the *hodograph* of the particle.

For a particle which is moving uniformly along a straight line the velocity is constant both as to magnitude and direction. Its hodograph, therefore, is merely a point. For a particle which is moving with constant speed in a circle, the magnitude of the velocity is constant but, as the direction is always changing, the hodograph, too, is a circle. The hodograph of a ball which describes a parabola under the action of gravity is a straight line. The hodograph of a planet in its motion about the sun is an eccentric circle, etc.

33. Accelerations are Vectors.—Let $\mathbf{V}_1$ (Fig. 22) be the velocity of the particle at the instant t, and let $\mathbf{V}_2$ be the velocity at the instant $t + \Delta t$. The change in the velocity during the interval Δt is

$$\mathbf{V}_3 = \mathbf{V}_2 - \mathbf{V}_1,$$

and the average rate of change is $\mathbf{V}_3/\Delta t$, which also is a vector with the same direction as $\mathbf{V}_3$ but of different magnitude. It will be relatively much longer if Δt is very small with respect to unity.

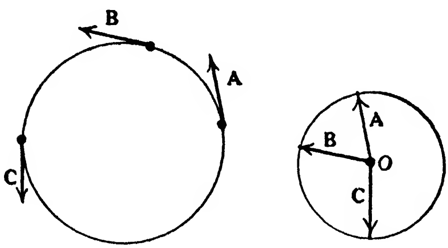


FIG. 21.

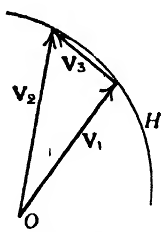


FIG. 22.

If the instant t be kept fixed and the interval Δt be diminished the terminus of the vector $\mathbf{V}_2$ will move along the hodograph as $\mathbf{V}_2$ approaches $\mathbf{V}_1$. The magnitude V_3 diminishes, but $V_3/\Delta t$ tends toward a limit. The direction of $\mathbf{V}_3$ also tends toward a limit which is evidently the direction of the tangent to the hodograph at the terminus of $\mathbf{V}_1$. Thus the vector $\mathbf{V}_3/\Delta t$, the average rate of change of the velocity during the interval Δt, tends toward a limit which may be denoted by $\mathbf{A}$, and which is called *the acceleration of the particle at the instant t.*

It will be observed that the acceleration of the particle $\mathbf{A}$ is simply the velocity of the terminus of the velocity vector in the hodograph. Since velocities are vectors, so also are accelerations. It is not necessary, therefore, to repeat the arguments which were used for velocities.

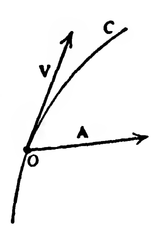


FIG. 23.

34. Résumé.—If a particle is moving along a curve C (Fig. 23), at each point O of the curve it has a velocity $\mathbf{V}$ which is tangent to the curve and an acceleration $\mathbf{A}$ which is directed toward the concave side of the curve. The magnitude of $\mathbf{V}$ is the speed at O. The magnitude and direction of the acceleration $\mathbf{A}$ depend upon the speed of the particle V and the curvature of C at O.

35. Constant Acceleration for Straight-line Motion.—The simplest case of accelerated motion is the motion of a particle in a straight line. If the change in the speed is proportional to the time elapsed, the acceleration is constant; for, the direction of the acceleration is in the line of motion, which is constant, and its numerical value is

$$A = \frac{v_2 - v_1}{t_2 - t_1}, \tag{1}$$

where v_1 is the speed at the instant t_1, and v_2 is the speed at the instant t_2, which also, by hypothesis, is constant.

36. Variable Acceleration for Straight-line Motion.—If the quotient (Eq. (35.1)) is not constant, that is, if its value depends upon the segment chosen, then the acceleration is the limit of A in Eq. (35.1) for decreasing values of the interval $(t_2 - t_1)$, that is

$$A = \operatorname*{limit}_{t_2 = t_1} \frac{v_2 - v_1}{t_2 - t_1} = \frac{dv}{dt}. \tag{1}$$

But since (Eq. (26.1)), for straight-line motion,

$$v = \frac{dx}{dt},$$

Eq. (1) becomes

$$A = \frac{dv}{dt} = \frac{d^2x}{dt^2}.$$

Hence, if the position of a particle which moves along a straight line is given as a function of t, the velocity and acceleration can be obtained by differentiations.

37. Example.—The position of a particle is given by the formula

$$x = a \cos nt.$$

Find its velocity and acceleration at any instant.

By differentiation, it is found that

$$v = \frac{dx}{dt} = -an \sin nt = \pm n\sqrt{a^2 - x^2}$$

and

$$A = \frac{d^2x}{dt^2} = -an^2 \cos nt = -n^2x.$$

If $0 < t < \pi/n$, the particle is moving toward the left (according to the usual conventions) and the velocity is negative. If $\pi/n < t < 2\pi/n$, the particle is moving toward the right and the velocity is positive.

If the particle is on the right side of the origin, the acceleration is directed toward the left. If it is on the left side of the origin, the acceleration is directed toward the right. Thus, whichever side the particle may be on, the acceleration is always directed toward the origin.

38. The Acceleration of Gravity.—Experiments, carefully conducted, show that the numerical value of the acceleration of all bodies freely rising or falling in a vacuum near the surface of the earth is approximately 32.2 feet per second per second. That is, if a body is dropped from rest in a vacuum, at the end of the first second its speed will be 32.2 feet per second; at the end of the second second its speed will be 64.4 feet per second, and so on. In general, denoting by v the velocity (positive if directed upward) measured in feet per second, by t the number of seconds which have elapsed from the initial instant, by v_0 the velocity at the initial instant (which may be positive, negative, or zero) then

$$v = v_0 - gt,$$

where $g = 32.2$. The corresponding vector equation is

$$\mathbf{v} = \mathbf{v}_0 + \mathbf{g}t,$$

where $\mathbf{g}$ is a vector directed towards the earth and represents the velocity change in one second. The constant g is called the *acceleration of gravity.*

For heavy bodies, such as a stone, and for speeds that are not too high, the resistance offered by the air is usually not important and can be neglected, the motion being regarded as occurring in a vacuum. For light bodies, such as a bit of paper or a fluffy feather, the resistance offered by the air is very important, and the motion of such bodies in the air is very different from what it would be in a vacuum.

39. Uniform Circular Motion.—A simple and very common type of motion is the motion of a particle in a circle with constant speed. It is desired to examine the nature of the acceleration in this case.

Let *EFB*, in Fig. 24, be an arc of the circle of which *O* is the center. Let *E* and *B* be any two points on the circle, and let $\mathbf{v}_1$ be the velocity of the particle at *E*, and $\mathbf{v}_2$ be the velocity of the particle at *B*. Since the speed is constant,

$$v_1 = v_2 = v.$$

The change in the velocity is

$$\mathbf{v}_3 = \mathbf{v}_2 - \mathbf{v}_1.$$

If $\mathbf{v}_1$ and $\mathbf{v}_2$ have their origin at E, and C is the terminus of $\mathbf{v}_1$ and D is the terminus of $\mathbf{v}_2$, then the parallelogram $ECBD$ is a rhomb. Its diagonals bisect each other perpendicularly, and since BE is a chord of the circle, the line of CD passes through the center of the circle, wherever the points E and B may be.

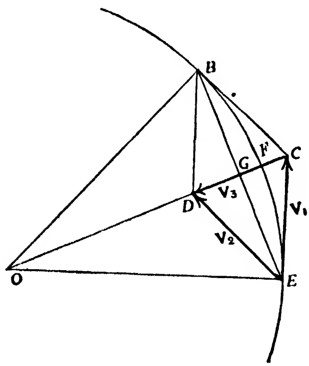


FIG. 24.

Suppose the point B is close to the point E and that the particle moves from E to B in the interval of time dt. In the limit, as B tends toward E,

$$EB = v\,dt, \qquad \mathbf{v}_3 = \overrightarrow{CD} = \mathbf{A}\,dt.$$

Also, if G is the center of the rhomb,

$$\frac{GD}{DB} = \frac{BG}{BO}, \quad \text{or} \quad \frac{1}{2}CD = \frac{1}{2}\cdot\frac{EB}{BO}\cdot DB.$$

Let a be the radius of the circle and A the magnitude of the acceleration. The equation

$$CD = \frac{EB}{BO}\cdot DB$$

becomes

$$A\,dt = \frac{v}{a}\cdot v\,dt,$$

or

$$A = \frac{v^2}{a}.$$

Hence, the acceleration is always directed toward the center of the circle; its magnitude is constant and is equal to the square of the speed divided by the radius of the circle (see Sec. 34).

If the angular speed be denoted by ω then

$$v = a\omega \qquad \text{and} \qquad A = a\omega^2,$$

which is another form of this very useful formula.

40. Vector Equations.—Let $\mathbf{i}$ be a unit vector along the x-axis, and $\mathbf{j}$ a unit vector along the y-axis. Let the particle be at the extremity of a radius which makes an angle θ with the x-axis.

Then the vector equations for the velocity and acceleration of a particle in uniform circular motion are

$$\mathbf{v} = v(-\sin\theta\cdot\mathbf{i} + \cos\theta\cdot\mathbf{j}),$$

$$\mathbf{A} = \frac{v^2}{a}(-\cos\theta\cdot\mathbf{i} - \sin\theta\cdot\mathbf{j}).$$

Problems III

1. A stone dropped over a cliff strikes the ground in 5 sec. With what speed did it strike and how high was the cliff? *Ans.* 161 ft. per second; 402.5 ft.

2. The brakes are applied to a train running 60 miles per hour. Assuming the retardation to be constant and equal to 4 ft. per second per second, how long does it take to bring the train to rest, and how far will the train go? *Ans.* 22 sec.; 968 ft.

3. A brick sliding on the sidewalk with a speed of 12 ft. per second comes to rest after sliding 24 ft. What is the retardation, assuming it to be constant? *Ans.* 3 ft. per second per second.

4. If the acceleration of a train is .3 ft. per second per second, how long will it take to increase its speed from 15 miles per hour to 60 miles per hour? *Ans.* 3 min. 40 sec.

5. What is the acceleration of gravity, (*a*) when the units are centimeters and seconds, (*b*) miles and minutes? *Ans.* (*a*) 981.5; (*b*) 21.95.

6. Five stones in succession are dropped over a cliff 1 sec. apart. What is the distance between the stones 2 sec. after the last one is dropped? *Ans.* 16.1 ft. multiplied by 11, 9, 7, and 5, respectively.

7. The speed of a particle in uniform circular motion is 4 ft. per second. The radius of the wheel is 2 ft. What is the acceleration? *Ans.* 8 ft. per second per second.

8. A windmill 12 ft. in diameter turns 40 times per minute. What is the acceleration of a point on the rim of the wheel? *Ans.* $32\pi^2/3$ ft. per second per second.

9. A and B are two points on a spoke of a wheel that is turning with uniform angular speed. Show that the motion of A with respect to B is uniform circular motion.

10. A horizontal disk is spinning uniformly and falling freely in a vacuum. Write the expression for the acceleration vector of a particle of the disk. *Ans.* $\mathbf{A} = -a\omega^2\cos\theta\cdot\mathbf{i} - a\omega^2\sin\theta\cdot\mathbf{j} - g\mathbf{k}$, where a is the distance of the particle from the center of the disk, ω is the angular speed, and $\mathbf{k}$ is a unit vector directed upward and perpendicular to $\mathbf{i}$ and $\mathbf{j}$.

11. What is the acceleration of a particle on the equator, assuming that the diameter of the earth is 7927 miles and that there are 86,164 sec. in a sidereal day? *Ans.* $g/289.4 = 0.1113$ ft. per second per second.

12. Assuming that the earth is perfectly rigid, what would be the length of the day if the acceleration at the equator were just equal to g? *Ans.* $1^h\ 24^m\ 27^s = 1/17$ of present day (nearly).

13. Assuming that the orbit of the moon is a circle of the radius of 238,000 miles and that its period is 27.3 days, what is the acceleration of the moon towards the earth as compared with g? *Ans.* $g/3611$.

14. What is the acceleration of the earth toward the sun, if the distance of the sun is 92,900,000 miles and the year is 365 1/4 days? *Ans.* $g/1656$.

15. A particle A moves uniformly across the diameter of a circle from left to right with the speed V. A particle B moves uniformly around the circle counterclockwise with the speed V. If the radius to B makes an angle θ with the given diameter, what is the velocity of B with respect to A?

Ans. $\mathbf{V} = V[-(1 + \sin\theta)\mathbf{i} + \cos\theta \cdot \mathbf{j}]$.

16. Prove that the acceleration vector is always directed toward the concave side of the curve of motion (Sec. 34). Suggestion: Use the hodograph.

17. By means of the hodograph, show that in uniform circular motion the acceleration is always directed toward the center of the circle.

CHAPTER IV

MASS AND FORCE

41. The Nature of the Mass.—The mass of a body corresponds to the idea of the quantity of matter which the body contains. It is independent of the body's position, of its state of motion, and of the forces which may be acting on the body. How the quantity of matter contained in a body shall be measured is not a simple question. If two bricks are physically alike, it is easy to see that taken together they contain twice as much matter as either one of them. But how can it be determined that they are "physically alike"? Given a cube of clay and a cube of lead of the same size, what is the ratio of their masses? If, as modern theories would indicate, all substances are merely different arrangements of electrons which are all alike then the ratio of the masses is merely the ratio of the number of electrons which the bodies contain. There are difficulties, however, not the least of which is the uncertainty that all electrons are alike. Nature does not seem to produce any two things which are just alike. The question will be passed over for the moment and taken up again in Sec. 47. Evidently mass is a scalar quantity.

42. Definition of a Particle.—A particle is a portion of matter which is sufficiently small for our purposes. It may be the size of the earth, or it may be the size of an electron, depending upon circumstances. Ordinarily it is desired to exclude the notion of its rotation, and also it is desired to be able to assert with sufficient exactness that it is located "at a certain point." The statement that "a particle is a material point" would be ideal, if only it had any sense.

43. Definition of Momentum.—The momentum of a particle is the product of its mass and its velocity,

$$\text{momentum} = \text{mass} \times \text{velocity}.$$

It corresponds to the notion "quantity of motion." Since velocity is a vector and mass is a scalar, momentum is a vector.

44. Undefined Concepts.—No effort is made to define time, force, or mass. It is assumed that the student knows their meaning from the way they are used. But it is important that it should be known how to measure them. This is accomplished by means of three postulates known as Newton's laws of motion.

45. Newton's Laws of Motion.—The science of mechanics rests primarily upon three statements:

I. *Every particle continues in its state of rest or uniform motion in a straight line unless it is acted upon by some exterior force.*

II. *The rate of change of momentum of a particle is proportional to the force impressed upon it, and is in the direction in which the force is acting.*

III. *To every action there is an equal and oppositely directed reaction.*

46. Discussion of the First Law.—If a body slides along a rough horizontal board with a certain initial speed, it will travel a certain distance and then will stop. If the same experiment be tried with a smoother board and a smoother body, but with the same initial speed, the body will again stop, but it will have traveled farther than the previous one. If a third body be mounted on wheels to still further reduce the friction, it will travel much farther than the former two, but eventually it, too, will stop. From such simple experiments Galileo inferred that if the resistance of the air and the force of friction could be eliminated altogether, the body would not stop at all, but would continue to move along a straight line and would eventually pass any assigned point of the line that was in the direction of its motion, however remote the point might be. This was not exactly an induction; rather, it was an inference drawn from his observations.

There was no basis for the inference that the motion along the straight line was *uniform*, because uniform motion means equal distances in equal times. As no means of measuring time has, as yet, been assigned, there is no criterion for telling when two intervals of time are equal. Thanks to the Euclidean postulate, it is known when two intervals on a straight line are equal. For the purposes of mechanics, an analogous postulate is needed to tell when two intervals of time are equal.

Imagine a particle moving along a straight line subject to no force whatever, and consider two intervals of equal length along

that line. Geometrically, there is nothing to distinguish one interval from the other save the difference of location; and mechanically there is nothing to distinguish the passage of the particle across the two intervals, save that one is subsequent to the other. It is natural to assume, therefore, that the corresponding intervals of time are equal. Accordingly, it is made a part of the postulate that this is so by the introduction of the word "uniformly." It follows then from the first law that the distance traveled by the particle along the straight line is proportional to the interval of time during which it was traveling; and this is the postulate by means of which time is measured.

47. Discussion of the Second Law.—Since momentum is a vector, the rate of change of momentum, that is, force, also is a vector and is equal to mass times acceleration, the mass being a mere constant and acceleration being the rate of change of velocity. Accordingly, the second law states that

$$\text{Force} = \text{mass} \times \text{acceleration}.$$

It must not be supposed that this expression tells what force is. It tells merely how forces are measured, namely, by the accelerations with which they are associated. Force, however, is always thought of as a push or a pull.

It will be observed that Newton's second law says nothing about causation. Since the force and the acceleration are simultaneous, there is no more reason for asserting that force is the cause of the acceleration than for asserting that acceleration is the cause of force. The fact that force is commonly spoken of as the cause of acceleration merely shows that in the order of our thoughts, force is commonly placed before acceleration. In the philosophical sense, nothing is known about causation. The common language, however, is convenient, and will do no harm if it is properly understood.

If two forces act successively upon the same mass, then the two forces are proportional to the accelerations produced; and thereby the forces are measured if the mass is known. The second law of motion would serve as a means of measuring masses if we had some independent means of knowing when two forces are equal; for, if two different masses are acted upon by forces which are equal, the accelerations can be measured and then the masses are inversely proportional to the accelerations. It is natural to suppose, for example, if a given elastic string is

stretched to half again its original length (supposing such a stretch to be possible) that the force required to stretch it the first time is the same as that required the second time, but though this is commonly done it is easy to see that a new postulate is used and should be stated explicitly.

In a sense the first law of motion is a corollary of the second law; for, if the rate of change of momentum is zero, the second law reduces to the first. But the rate of change of momentum cannot be measured until a criterion for the measurement of time has been established, and for this the first law is necessary.

The first two laws were known to Galileo, although he did not state them explicitly.

48. Discussion of the Third Law.—The words "action" and "reaction" are here to be understood as forces. The law states that in nature forces always occur in pairs, the two components of which are equal in magnitude and opposite in direction. If the particle A acts upon the particle B, then the particle B also acts upon A equally but in the opposite direction. A force, so to speak, cannot hitch itself to nothing.

The third law of motion is due to Newton. The law of gravitation also is due to Newton, and on account of its importance it will be given here.

49. The Law of Gravitation.—As stated by Newton:

Every particle in the universe attracts every other particle with a force which is directly proportional to the product of the masses of the particles, and inversely proportional to the square of the distance between them.

$$f = k^2\frac{m_1m_2}{r^2},$$

where f is the magnitude of the force, m_1 and m_2 the masses of the particles, r the distance between the particles, and k^2 is a constant which depends upon the units which are employed. In the English system of units (Sec. 51) $k^2 = 1.068 \times 10^{-9}$. In the c.g.s. system of units $k^2 = 6.66 \times 10^{-8}$. It will be observed that the law of gravitation is stated only for particles, but it can be shown that it holds also for uniform spheres, the distance being measured from the centers of the spheres.

50. Weight and the Acceleration of Gravity.—As a particular consequence of the law of gravitation, the earth attracts all bodies near its surface with a force which is called the *weight* of the body.

The acceleration toward the earth due to the attraction of the earth is called the *acceleration of gravity* and is denoted by the letter g, so that, if w is the weight of the body and m is its mass, by Newton's second law,

$$w = mg, \qquad g = 32.174 - 0.085 \cos 2l,$$

where l is the latitude. This is the value of the acceleration of gravity for an observer who is at rest with respect to the surface of the earth at the place where the acceleration is measured.

If the earth were a sphere and not rotating and g_0 were the corresponding value of the acceleration, it would follow upon applying the law of gravitation that,

$$mg_0 = k^2\frac{mE}{R^2},$$

whence

$$g_0 = \frac{k^2E}{R^2},$$

where E is the mass of the earth and R is its radius.

The rotation of the earth does not affect a body which is freely falling near its surface, but the oblateness of the earth does affect it. Denoting the acceleration of gravity for a freely falling body by G, it is found that

$$G = 32.225 - 0.026 \cos 2l.$$

This would be the value of the acceleration for an observer at rest relative to the center of the earth.

The effect of the oblateness of the earth upon the attraction near its surface belongs to the theory of the potential which lies beyond the scope of this work; but the effect of the rotation of the earth will be considered in Sections 265 and 360.

Since, by the law of gravitation, the force of attraction of the earth upon any body is proportional to the mass of the body, it follows that the acceleration of gravity g is the same for all bodies at the same place on the earth, and therefore, *the mass of a body is proportional to its weight at any given place.*

51. Systems of Units.—Among English speaking people, there are two systems of units in common use.

THE ENGLISH SYSTEM

In the old English system

the unit of length is the foot,

the unit of time is the second,
the unit of mass is a mass
which weighs one pound.

The standard of mass is a certain mass of platinum which is kept in London and which weighs one pound in London.

The standard of length is a certain bronze bar on which two lines are engraved one yard (equal to three feet) apart. The yard is defined to be the distance between these two lines when the bar is at a temperature of 62°F. Accurate copies of these standards are kept in the United States Bureau of Standards at Washington.

By the second law of motion, force is proportional to mass times acceleration. It is convenient to choose the units so that force *equals* mass times acceleration. If gravity acts upon a one-pound mass, it follows that

$$\text{force} = 1 \times 32.2 = 32.2 \text{ units of force},$$

and that this force is equal to the weight of a one-pound mass. The unit of force is called a *poundal* and, therefore,

$$\text{the poundal} = \frac{\text{the weight of a one-pound mass}}{32.2}$$

and is approximately equal to the weight of one-half an ounce. A constant force of one poundal acting on a mass of one pound for one second will generate a speed of one foot per second. The poundal is called the *absolute* unit of force.

In statics it is usually more convenient to use the weight of a one-pound mass, namely one pound, as the unit of force, or even the weight of a ton mass. This is a *gravitational* unit of force. But when motion is involved, the unit of force is the poundal and the unit of mass is the mass of a one-pound weight.

A mass of one pound and a weight of one pound are very different concepts. Mass is independent of position while the weight is not.

THE SCIENTIFIC OR C.G.S. SYSTEM

The second system of units is the scientific system, since it is used by scientists the world over. In this system

the unit of length is the centimeter,
the unit of time is the second,
the unit of mass is the gram.

The gram is defined as the mass of one cubic centimeter of pure water at a temperature of 4°C. The unit of force is the *dyne*, so that a force of one dyne acting for one second upon one gram will generate a speed of one centimeter per second.

Between these two systems of units the following relations exist:

$$
\begin{aligned}
1 \text{ foot} &= 30.48 \text{ centimeters},\\
1 \text{ pound} &= 453.59 \text{ grams},\\
1 \text{ poundal} &= 13{,}825.5 \text{ dynes}.
\end{aligned}
$$

Since the unit of time is the same in the two systems, the acceleration of gravity in the c.g.s. system is $32.18 \times 30.48 = 981$ centimeters per second per second, and the weight of one gram is equal to 981 dynes.

CHAPTER V

WORK AND ENERGY

52. Definition of Work.—If a man drags a weight along a level surface, or if he raises a weight against the force of gravity, he is conscious of doing *work*. In the first case, assuming the frictional resistance to be constant, the amount of work done is proportional to the resistance which the surface offers and the distance through which the weight is dragged. In the second case the work done is proportional to the weight raised and to the height through which it is raised. In each of these cases the resisting force is constant and its direction is the same as that of the motion of the body. Under these conditions then,

$$\text{work} = \text{force} \times \text{displacement}.$$

If the unit of force is one pound and the unit of length is one foot, the unit of work is one *foot-pound*. If the unit of force is one poundal, the unit of work is one *foot-poundal*. One foot-pound is equal to 32.2 foot-poundals.

If the unit of force is one dyne and the unit of length is one centimeter, the unit of work is one *erg*. One foot-poundal is equal to 421,400 ergs. Since the erg is a very small unit of work, it is sometimes convenient to have a larger unit which is called the *joule*. One joule is equal to ten million (10^7) ergs.

53. The Force and the Displacement are Oblique.—If the displacement is perpendicular to the force which is acting, then there is no work done, *e.g.*, moving a body along perfectly smooth ice against gravity. Of course, perfectly smooth ice does not exist in practice. There is always a certain amount of friction, and the work done against this friction is not zero. The friction is in the line of the displacement, however, and is not perpendicular to it. The work done *against gravity* is zero.

If the force **F** makes an angle θ with the displacement **D**, it can be resolved into two components, $\mathbf{F}_1$ in the line of the displacement and $\mathbf{F}_2$ perpendicular to this line. The second component does no work, since it is perpendicular to the displacement. The

work done by the first component is the total work which is done by **F**; hence,

$$W = F_1D = FD_1 = FD \cos \theta.$$

Stated in words, *the work done by a force is the product of the magnitudes of the displacement and the component of the force in the direction of the displacement;* or

The work done by a force is equal to the product of the magnitudes of the force and the component of the displacement in the direction of the force; or,

The work done by a force is equal to the product of the magnitudes of the force and the displacement and the cosine of the angle between them.

54. Positive and Negative Work.—If the displacement is in the direction of the force which is acting, the work done by the force is positive. If the displacement is in the opposite direction, the work done *by the force* is negative, that is, work has been done *against* the force. Thus, if a weight w is raised through a height h, the work done *by gravity* is $W = -wh$ and, if it is lowered through the same distance, the work done *by gravity* is $W = +wh$. Hence, the total work done *by gravity* on a weight which is raised and then lowered to its original position is zero.

55. Vector Products.—Both the force **F** and the displacement **D** are vectors. These vectors enter symmetrically in the expression for the work done; nevertheless, work is manifestly not a vector. It is an example of what is called, in the algebra of vectors, *the scalar product of two vectors*, and is expressed by the formula

$$W = \mathbf{F} \cdot \mathbf{D} = FD \cos \theta.$$

It is also called *the dot product.*

There exists also a second kind of multiplication of vectors called *the vector product of two vectors* or *the cross-product*,

$$\mathbf{A} = \mathbf{F} \times \mathbf{D},$$

where the direction of **A** is perpendicular to the plane of **F** and **D** and, in magnitude,

$$A = FD \sin \theta.$$

There will be an occasion in Sec. 133 to notice an example of this second type of multiplication.

56. Resolution of Both Force and Displacement.—Let the force and the displacement both be resolved into their components with respect to any rectangular set of axes, and let

$$\mathbf{F} = \mathbf{X} + \mathbf{Y} + \mathbf{Z},$$
$$\mathbf{D} = \mathbf{x} + \mathbf{y} + \mathbf{z}.$$

The work done by the force $\mathbf{F}$ in the displacement $\mathbf{x}$ is xX, for the components $\mathbf{Y}$ and $\mathbf{Z}$ are perpendicular to $\mathbf{x}$ and, therefore, do no work. Similarly, the work done by the force $\mathbf{F}$ in the displacement $\mathbf{y}$ is yY, and in the displacement $\mathbf{z}$ it is zZ. Hence, the total work done in the displacement $\mathbf{D}$ is

$$W = \mathbf{F} \cdot \mathbf{D} = FD \cos \theta = xX + yY + zZ. \qquad (1)$$

57. Example in Which the Force is Constant.—A weight of w pounds is dragged up a smooth (frictionless) plane which makes an angle α with a horizontal plane through a distance r. How much work is done?

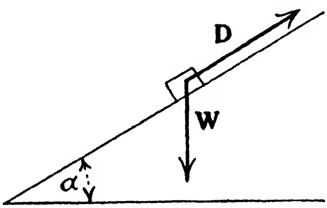


FIG. 25.

The component of the weight (force) in the line of the displacement $\mathbf{D}$ is $w \sin \alpha$ pounds. Hence, the work done is $wr \sin \alpha$ foot-pounds. But since $r \sin \alpha = h$, the height through which the weight is raised, the work done is wh foot-pounds and, as this expression is independent of α, it makes no difference how far the body is moved horizontally; it is only the height through which it is raised that counts.

58. The Work Done by a Variable Force.—If the force varies from point to point either in magnitude or direction, or both, the Eq. (56.1) will hold only for an infinitesimal displacement dx, dy, dz. Therefore only an element of work is obtained, *viz.*,

$$dW = F \cdot ds \cdot \cos \theta = Xdx + Ydy + Zdz. \qquad (1)$$

The total work done is the sum of all the elements of work, that is,

$$W = \int F \cos \theta \cdot ds = \int Xdx + \int Ydy + \int Zdz,$$

the integrals being taken along the path of motion.

59. Example in Which the Force is Variable.—How much work is required to raise a one-pound weight from the surface of the earth to the distance of the moon?

According to the law of gravitation, the force varies inversely as the square of the distance from the center of the earth. Let the radius of the earth expressed in feet be R, and let the distance of the moon be $60R$. If the unit of force is one pound, then at a distance r from the center of the earth, the magnitude of the force is

$$f = \frac{R^2}{r^2} \text{ pounds.}$$

The work done in an infinitesimal displacement dr is

$$dW = \frac{R^2}{r^2}dr \qquad \text{foot-pounds,}$$

and, therefore,

$$W = \int_R^{60R} \frac{R^2}{r^2}dr = \frac{59}{60}R \qquad \text{foot-pounds.}$$

Taking the radius of the earth to be $R = 20{,}908{,}800$ feet, it is found that

$$W = 20{,}560{,}320 \text{ foot-pounds.}$$

60. The Work Done in Stretching an Elastic String.—It is required to find the work done in stretching an elastic string of natural length l from a length a to a length b.

It is found from experiments that the tension of a stretched string is proportional to the amount of the stretch; that is, if x is the stretched length of the string, T is the tension, and k is a factor of proportionality,

$$T = k(x - l).$$

Hence, the work done in stretching the string from a to b is

$$\begin{aligned} W &= \int_a^b T dx = k\int_a^b (x - l)dx \\ &= k\left(\frac{a+b}{2} - l\right)(b - a) = T_m(b - a), \end{aligned}$$

where T_m is the tension of the string at the point midway between $x = a$ and $x = b$.

61. Work Represented by an Area.—If the displacement is measured along the x-axis and the force $F(x)$ is a function of x, the work done in moving the body from a to b against the force is

$$W = \int_a^b F(x)dx.$$

This integral evidently is the area included between the curve

$$y = F(x),$$

the x-axis, and the two ordinates at $x = a$ and $x = b$. In the example of the preceding section,

$$F(x) = k(x - l),$$

that is, a straight line. The work done in stretching the elastic string is the area of the trapezoid $ABCD$, which is evidently equal to the product of the middle ordinate T_m by the base $\overline{AB}$.

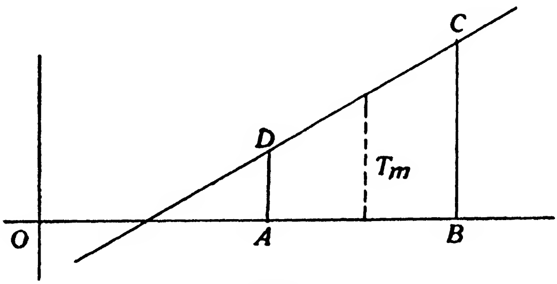


FIG. 26.

62. Power.—The rate at which work is being done is called *power*. The ordinary unit of power is the horsepower, which was defined by James Watt to be 33,000 foot-pounds per minute, or 550 foot-pounds per second. It requires an unusually good horse to work at this rate continuously, but even a poor horse can work at this rate for a short interval. In electrical measurements the term *watt* is used, one watt being equal to 10^7 ergs (one joule) per second. It is the rate of working in a circuit when the e.m.f. is one volt and the current is one ampere. One horsepower is equal to 746 watts, approximately.

The rate at which work is being done is evidently equal to the product of the force and the rate of displacement, if the displacement is in the line of the force, that is,

$$\text{power} = \text{force} \times \text{speed}.$$

If the displacement is not in the line of the force, the general formula, derived from Eq. (58.1), is

$$\frac{dW}{dt} = F \cdot \frac{ds}{dt} \cdot \cos\theta = X\frac{dx}{dt} + Y\frac{dy}{dt} + Z\frac{dz}{dt}.$$

63. Example.—At what rate can a steam roller of 30 hp. and of a weight of 3 tons move along a road if the horizontal resistance to its motion is equal to its own weight?

Since the horizontal force is 6000 lb., it requires 6000 ft.-lb. of work to move the roller 1 ft. If the amount of work available is 30 hp., or 990,000 foot-pounds per minute, it can move

$$\frac{990{,}000}{6000} = 165 \text{ ft. per minute,}$$

or 1 7/8 miles per hour.

64. The Force Function.—Returning to the formula for the element of work (Eq. (58.1)),

$$dW = Xdx + Ydy + Zdz,$$

the components of the force X, Y, and Z are functions of the coordinates x, y, and z; that is, the force acting depends upon the position of the body. They may also depend upon the components of the velocity,

$$\dot{x} = \frac{dx}{dt}, \qquad \dot{y} = \frac{dy}{dt}, \qquad \dot{z} = \frac{dz}{dt},$$

as is the case when friction enters. They may also depend upon the time t.

There is a very important class of cases, such as gravitation for example, in which the force depends upon the position of the body alone. Suppose X, Y, and Z are *single-valued* functions of x, y, and z which do not contain $\dot{x}$, $\dot{y}$, $\dot{z}$, t; and furthermore, that there exists a function, $U(x, y, z)$, such that

$$X = \frac{\partial U}{\partial x}, \qquad Y = \frac{\partial U}{\partial y}, \qquad Z = \frac{\partial U}{\partial z}.$$

Then, for reasons that are obvious, $U(x, y, z)$ is called the *force function*, or sometimes, for reasons that are not quite so obvious, the *potential function*. If these conditions are satisfied, the formula for the element of work done by the forces becomes

$$dW = \frac{\partial U}{\partial x}dx + \frac{\partial U}{\partial y}dy + \frac{\partial U}{\partial z}dz = dU,$$

which is an exact differential. It can be integrated from any point x_1, y_1, z_1, to any other point x_2, y_2, z_2, without any knowledge of the path (Sec. 58) along which the motion occurred, provided U also is a single-valued function of x, y, z. Thus

$$W = U(x_2, y_2, z_2) - U(x_1, y_1, z_1), \tag{1}$$

whatever the path may have been. *The work done is independent of the path.*

If X, Y, and Z are single-valued functions of x, y, and z, independent of $\dot{x}$, $\dot{y}$, $\dot{z}$, t, the necessary and sufficient condition that a force function $U(x, y, z)$ exists is that

$$\frac{\partial X}{\partial y} = \frac{\partial Y}{\partial x}, \qquad \frac{\partial Y}{\partial z} = \frac{\partial Z}{\partial y}, \qquad \frac{\partial Z}{\partial x} = \frac{\partial X}{\partial z}.$$

65. The Force Function is Not Single Valued.—If the force function is not single valued, Eq. (64.1) does not have a precise meaning, since there would be doubt as to which value of U to take. Suppose, for example, for the motion of a particle,

$$U = \tan^{-1}\frac{y}{x},$$

so that

$$X = -\frac{y}{x^2 + y^2}, \qquad Y = +\frac{x}{x^2 + y^2}, \qquad Z = 0.$$

Notwithstanding U is multiple valued, X, Y, and Z are single valued at all points, except the origin where X and Y are not defined. The element of work is

$$dW = \frac{xdy - ydx}{x^2 + y^2},$$

or, if

$$x = r\cos\theta, \qquad y = r\sin\theta,$$

in polar coordinates,

$$dW = d\theta.$$

Since this expression does not depend upon dr, no work is done if the particle is moved along a radius vector, either away from or toward the origin. The work done is equal to the angle about the origin through which the particle turns. It is equal to 2π for each complete circuit about the origin. Hence, the work done in moving the particle from A to B depends upon the number of circuits about the origin made in the path from A to B.

Such uncertainties do not arise if U is single valued. If X, Y, and Z are not single valued, the physical situation is not uniquely defined.

66. Energy.—It is commonly stated that *energy* is the capacity for doing work, or that *work is the measure of energy.* In reality, energy cannot be defined. Along with space and time it is one

of the fundamental concepts in terms of which an attempt is made to interpret nature and, therefore, it lies beyond the reach of a definition. Energy is recognized in two forms, namely, *potential*, the energy which a body possesses by virtue of the position which it occupies, and *kinetic*, the energy which a body possesses by virtue of the fact that it is moving. Energy is either kinetic or potential; we know of no other form: and it is assumed that energy can be neither created nor destroyed.

67. Conservative and Dissipative Forces.—Let W be the work required to move a system of bodies from a certain configuration A to a certain other configuration B *against* the system of forces F which are acting upon the system of bodies. If the work done *by* the system of forces upon the system of bodies when it is returned from the configuration B to the configuration A also is equal to W, then the system of forces F is said to be *conservative*. The energy which was absorbed by the system in the displacement from A to B is restored, or given back, in the displacement from B to A. All gravitational forces, for example, are conservative.

If, on the other hand, the work done by the system of forces F in the return displacement from B to A is less than W, then F is said to be a *dissipative* system of forces. Friction and viscosity are typical dissipative forces. When such forces occur, a certain amount of the energy passes from the mass to the molecules where it is recognized as heat, which is radiated away, and some of it, doubtless, becomes submolecular; at any rate, it is not available in restoring the system of bodies from the configuration B to the configuration A.

Under no circumstances is the work done by the forces F in the return from B to A greater than the work done against them in the displacement from A to B. It is this principle which denies the possibility of what is commonly called *perpetual motion*. Since friction cannot be eliminated in any terrestrial experiment, no terrestrial mechanism can be made to run indefinitely without supplying it with energy from the outside.

68. Potential Energy.—Suppose that the system of forces is conservative and that there exists a force function U. If the function U were multiple valued, it would be possible to move the system of bodies from a configuration A around a certain circuit back to the configuration A in which U changed from one of its values at A to another one of its values at A. If it were necessary

to do work upon the system in order to make it go around this circuit, then, on account of the fact that the forces are conservative, work would be done by the system if the circuit were reversed. Since, by hypothesis, the system has returned to its initial configuration, it could go through the reversed circuit a second time and therefore do some more work, and so on. Perpetual motion would be possible and the delightful situation would exist of being able to get something out of nothing. Since this is manifestly impossible the force function U must be single valued for every physical situation.

The potential energy of a system of bodies in a configuration B with respect to a configuration A is the amount of work which must be done upon the system in order to bring it from A to B, and this is independent of the path. It will be observed that potential energy is always relative, that is, one configuration with respect to another.

Let m be the mass of a particle, x_0, y_0, z_0 its initial position, x, y, z its terminal position, and $U(x, y, z)$ the force function. Then the potential energy of its terminal position with respect to its initial position is

$$V(x, y, z) = U(x_0, y_0, z_0) - U(x, y, z),$$

since the work is done *against* the forces of the system. Its potential energy with respect to any other position x_1, y_1, z_1, will differ from this only by the constant value

$$U(x_1, y_1, z_1) - U(x_0, y_0, z_0).$$

From these relations, it is seen that the potential energy differs from the negative of the force function (Sec. 64) only by an additive constant. Therefore, if V is the potential energy, the components of force can also be written

$$X = -\frac{\partial V}{\partial x}, \qquad Y = -\frac{\partial V}{\partial y}, \qquad Z = -\frac{\partial V}{\partial z}.$$

Stated in words, *the x-, y-,* and *z-components of the force at any point are the values of the negative derivatives of the potential energy* with respect to x, y, and z, respectively, at that point.

69. Generalization.—More generally, it is true that *the negative derivative of the potential energy at any given point in any specified direction is the component in the assigned direction of the force acting at that point.*

In order to show this, let x_0, y_0, z_0 be the given point and let ξ, η, ζ be a set of rectangular axes with origin at x_0, y_0, z_0 and with the ξ-axis in the given direction. Then the coordinates of any point in the two systems are related by the equations

$$\begin{aligned} x &= x_0 + \alpha_1\xi + \alpha_2\eta + \alpha_3\zeta, \\ y &= y_0 + \beta_1\xi + \beta_2\eta + \beta_3\zeta, \\ z &= z_0 + \gamma_1\xi + \gamma_2\eta + \gamma_3\zeta. \end{aligned}$$

The coefficients α_i, β_i, and γ_i are the direction cosines of the angles between the various axes as indicated in the table:

	ξ	η	ζ
x	α_1	α_2	α_3
y	β_1	β_2	β_3
z	γ_1	γ_2	γ_3

Since ξ enters the expression for V only through the coordinates x, y, and z, it follows that

$$\frac{\partial V}{\partial \xi} = \frac{\partial V}{\partial x}\cdot\frac{\partial x}{\partial \xi} + \frac{\partial V}{\partial y}\cdot\frac{\partial y}{\partial \xi} + \frac{\partial V}{\partial z}\cdot\frac{\partial z}{\partial \xi};$$

and, since

$$\frac{\partial V}{\partial x} = -X, \qquad \frac{\partial V}{\partial y} = -Y, \qquad \frac{\partial V}{\partial z} = -Z,$$

$$\frac{\partial x}{\partial \xi} = \alpha_1, \qquad \frac{\partial y}{\partial \xi} = \beta_1, \qquad \frac{\partial z}{\partial \xi} = \gamma_1,$$

it follows that

$$-\frac{\partial V}{\partial \xi} = \alpha_1 X + \beta_1 Y + \gamma_1 Z.$$

Let the components of the force along the ξ-, η-, and ζ-axes be Ξ, H, and Z, respectively. Then since α_1, β_1, γ_1 are the cosines of the angles between the ξ-axis and the x-, y-, and z-axes, respectively, it follows by projection that

$$\Xi = \alpha_1 X + \beta_1 Y + \gamma_1 Z.$$

Therefore,

$$\Xi = -\frac{\partial V}{\partial \xi};$$

and similarly $$H = -\frac{\partial V}{\partial \eta}, \qquad Z = -\frac{\partial V}{\partial \zeta},$$

which proves the proposition.

observed that the kinetic energy vanishes with the speed and that it is never negative.

If f is a function of x alone, that is, it does not depend upon $\dot{x}$ or t, and is an integrable function of x, then there exists a function $V(x)$ such that

$$f(x) = -\frac{\partial V}{\partial x}.$$

In this event the forces acting are conservative, and $V(x)$ differs from the potential energy only by an additive constant. Hence, the integral of Eq. (1) can be written

$$\frac{1}{2}m\dot{x}^2 + V(x) = \text{constant}; \tag{3}$$

or, in words, *the sum of the kinetic and potential energies of the particle is constant.* This result, which is proved here in a very simple case, is characteristic of conservative systems, however complicated they may be. The energy equation (Eq. (3)) obviously does not exist for a dissipative system.

71. Dimensions.—When it is said that the line $\overline{AB}$ is of length a, what is meant is that a unit of length L has been chosen and that the length $\overline{AB}$ is a times the length L. That is

$$\overline{AB} = a \cdot L.$$

The unit of length having been adopted, the unit of area is L^2 and the unit of volume is L^3, that is, a square and a cube each of whose edges is equal to L. When it is said that the area of a rectangle, whose sides are aL and bL, is equal to ab, it is meant that the area is equal to ab times the unit of area L^2; and similarly, the volume of the parallelepiped whose edges are aL, bL, and cL is the product abc times the unit of volume L^3. From this explanation it will be clear what is meant when it is said that a line has the dimension L, an area has the dimension L^2, and a volume has the dimension L^3. A product of four lengths would have the dimension L^4, and so on. The quotient of a volume by an area is of the form $(aL^3)/(bL^2) = cL$, that is, its dimension is L.

In mechanics there are three fundamental units, length, time, and mass, denoted respectively by the letters L, T, and M, and therefore the mechanical concepts, velocity, acceleration, momentum, force, energy, power, etc., have dimensions which are expressible in these three units. Thus

$$
\begin{aligned}
\text{velocity} &= \text{length} \div \text{time} &&= LT^{-1},\\
\text{acceleration} &= \text{velocity} \div \text{time} &&= LT^{-2},\\
\text{momentum} &= \text{mass} \times \text{velocity} &&= MLT^{-1},\\
\text{force} &= \text{mass} \times \text{acceleration} &&= MLT^{-2},\\
\text{work (or energy)} &= \text{force} \times \text{length} &&= ML^2T^{-2},\\
\text{power} &= \text{work} \div \text{time} &&= ML^2T^{-3}.
\end{aligned}
$$

72. Homogeneity.—A mechanical quantity, which with respect to the fundamental units is of the form $M^iL^jT^k$, is said to be of dimensions i in mass, j in length, and k in time. Two quantities that are alike, that is, of the same mechanical nature, must have the same dimensions. By an elementary principle of arithmetic, only quantities that are of the same nature can be added together. From this principle it follows that the terms of an equation must all have the same dimensions in the fundamental units. That is, *the equations of mechanics are homogeneous in the fundamental units.* This principle is frequently of value for the detection of errors.

73. Changing the Units.—It is desired occasionally to translate the numerical value of a physical magnitude, given in one set of units, to its equivalent numerical value in another set of units. Let X be a mechanical magnitude; let L_1, M_1, and T_1 be one set of units, L_2, M_2, and T_2 be a second set of units, and, finally, let

$$
\begin{aligned}
X &= x_1L_1{}^iM_1{}^jT_1{}^k\\
\text{and}\qquad X &= x_2L_2{}^iM_2{}^jT_2{}^k,
\end{aligned}
\tag{1}
$$

where x_1 and x_2 are the numerical values for the two sets of units respectively. Evidently the first equation of (1) can be written

$$X = x_1\left(\frac{L_1}{L_2}\right)^i\left(\frac{M_1}{M_2}\right)^j\left(\frac{T_1}{T_2}\right)^k L_2{}^iM_2{}^jT_2{}^k;$$

and, on comparing this expression with the second equation of (1), it is seen that

$$x_2 = \left(\frac{L_1}{L_2}\right)^i\left(\frac{M_1}{M_2}\right)^j\left(\frac{T_1}{T_2}\right)^k x_1;$$

or, on setting

$$\lambda = \frac{L_1}{L_2}, \qquad \mu = \frac{M_1}{M_2}, \qquad \tau = \frac{T_1}{T_2},$$

the simpler expression

$$x_2 = \lambda^i\mu^j\tau^k \cdot x_1.$$

74. Example.—The value of the gravitational constant in astronomical units is $\log k^2 = 6.4712 - 10$. What is its value in c.g.s. units?

In the astronomical system, the unit of mass is the mass of the sun ($= 1.990 \times 10^{33}$ g.). The unit of length is the mean distance of the earth from the sun ($= 1.493 \times 10^{13}$ centimeters), and the unit of time is the mean solar day ($= 86{,}400$ sec.).

From the law of gravitation is derived, Sec. 49,

$$\text{force} = k^2 \frac{m_1 m_2}{r^2}$$

or, dimensionally,

$$MLT^{-2} = k^2 M^2 L^{-2}.$$

Hence, from the point of view of dimensions,

$$k^2 = M^{-1} L^{+3} T^{-2}.$$

Also,

$$\lambda = 1.493 \times 10^{13}, \quad \mu = 1.990 \times 10^{33}, \quad \tau = 86{,}400,$$
$$i = +3, \quad j = -1, \quad k = -2.$$

Therefore,

$$\log [\lambda^i \mu^j \tau^k] = 6.3504 - 10,$$

and the value of the $\log k^2$ expressed in c.g.s. units is

$$\log k^2 = 2.8216 - 10.$$

Problems IV

1. A man weighing 150 lb. runs up a flight of stairs 55 ft. high in 30 sec. How much power does he develop? *Ans.* 1/2 hp.

2. How fast can a team of 2 hp. draw a loaded wagon weighing 2 tons, if the resistance to rolling is one-twentieth of the weight of the load? *Ans.* 3 3/4 miles per hour.

3. How much power is required to pump 1000 gal. of water per minute from a well 110 ft. deep if a gallon of water weighs 8.34 lb. and half of the total work done is used in overcoming internal friction? *Ans.* 55.6 hp.

4. Three horsepower are required to drag a 220-lb. weight at the rate of 10 ft. per second along an inclined plane which makes an angle of 30° with the horizontal. What is the force of resistance due to friction? *Ans.* 55 lb.

5. From the relations between the fundamental units, show that 1 foot-poundal equals 421,400 ergs.

6. The speed of the earth in its orbit is 18.5 miles per second. What is its speed in astronomical units? *Ans.* 0.0172 A.U. per day.

7. The numerical value of the acceleration of the earth toward the sun in astronomical units is 0.0002959. What is its value in c.g.s. units? *Ans.* 0.592 cm. per second per second.

8. What horsepower is required to move a body with the speed v ft. per second if it is resisted by a force of p poundals? *Ans.* $pv/17{,}710$.

9. A steamship of 22,000 hp. travels 3300 miles in 6 days. What is the average resistance to the motion of the ship? *Ans.* 180 tons.

10. A locomotive pulls a train of 3000 tons up an incline of 1 ft. per 100 ft. at 5 miles per hour. What is the power of the engine if the resistance due to friction is 1/200 of the weight of the train? *Ans.* 1200 hp.

11. Fifty horsepower is transmitted from one shaft to another by means of a belt and pulleys. The linear speed of the belt is 250 ft. per minute. What is the difference in the tensions of the two sides of the belt? *Ans.* 6600 lb.

12. Multiply together by dot multiplication the expressions for **F** and **D** in Sec. 56, thus

$$\mathbf{F}.\ \mathbf{D} = \mathbf{x}\cdot\mathbf{X} + \mathbf{x}\cdot\mathbf{Y} + \mathbf{x}\cdot\mathbf{Z} + \cdots$$

Evaluate the individual terms and verify the result there given.

13. What is the potential energy of a particle, if the components of the force acting on it, expressed in pounds, are $6x$, $6y$ and $6z$? *Ans.* $-3r^2$ ft.-lb., if 1 ft. is the unit of length.

14. A uniform sphere of mass M attracts an exterior particle of mass m in accordance with the law of gravitation. Show that the potential energy of the particle in any position is

$$V = -k^2\frac{Mm}{r},$$

where r is the distance of the particle from the center of the sphere.

15. A particle is attracted toward a fixed center by a force which is directly proportional to its distance from the fixed center. If the magnitude of the force at a unit distance is j^2, show that the potential energy of the particle is

$$V = \frac{1}{2}j^2r^2.$$

16. An elastic string of length l ft. and of negligible weight stretched to twice its natural length exerts a pull of k lb. A weight of m oz. is attached to the end of the stretched string and the string is then released. What is the maximum speed imparted to the weight? *Ans.* $V = 4\sqrt{lkg/m}$ ft. per second.

17. A bullet with a speed of 1000 ft. per second will penetrate a block of wood 12 in. Assuming that the resistance to the bullet is constant, show that it will penetrate and emerge from a plank 2 in. thick with a speed of 913 ft. per second.

18. Two equal weights W_1 are supported by a string which passes over two frictionless pulleys at A and B in the same horizontal line. A third weight, $W_2 = 2W_1/\sqrt{3}$, is attached to the string half way between A and B, and allowed to drop from a state of rest. Show that it will descend until AW_2B is an equilateral triangle. What will happen after this? *Ans.* It will rise to its initial position.

19. The weight of a spider, which is hanging from the ceiling by its web, doubles the natural length of the web. Show how the spider can climb back to the ceiling with an expenditure of work which is only three-quarters of what would be required if the web were inelastic.

20. A rod of length $2l$ and weight w rests horizontally in an ellipsoidal bowl of which the semiaxes are a, b, and c. If the rod, always horizontal with its ends in contact with the bowl, is turned through an angle θ, show that its potential energy is

$$V = cw\left[\sqrt{\frac{a^2 - l^2}{a^2}} - \sqrt{1 - \frac{l^2 \cos^2 \theta}{a^2} - \frac{l^2 \sin^2 \theta}{b^2}}\right].$$

21. What speed must be imparted to a projectile which is fired vertically from the surface of the earth in order that it may never return? *Ans.* 6.95 miles per second.

CHAPTER VI

GEOMETRICAL CONCEPTS

75. Explanation.—In dealing with bodies of finite dimensions there are two notions which are constantly presenting themselves and which belong properly to the geometry of the bodies rather than to mechanics; but, inasmuch as they are indispensable for mechanics, they will be treated here briefly. The first of these concepts is the center of gravity and the second, moments of inertia.

I. THE CENTER OF GRAVITY

76. The Center of Gravity of a Discrete Set of Particles.—Suppose there is given a system of n particles; that the mass of the ith particle is m_i and that its coordinates with respect to a fixed set of axes are x_i, y_i, and z_i. The center of gravity of the system is a point the coordinates of which are the weighted mean of the corresponding coordinates of the particles, the weighting factors being the masses of the particles. If $\bar{x}$, $\bar{y}$, and $\bar{z}$ are the coordinates of the center of gravity, then

$$\bar{x} = \frac{\Sigma m_i x_i}{\Sigma m_i} = \frac{m_1 x_1 + m_2 x_2 + \cdots + m_n x_n}{m_1 + m_2 + \cdots + m_n},$$
$$\bar{y} = \frac{\Sigma m_i y_i}{\Sigma m_i} = \frac{m_1 y_1 + m_2 y_2 + \cdots + m_n y_n}{m_1 + m_2 + \cdots + m_n},$$
$$\bar{z} = \frac{\Sigma m_i z_i}{\Sigma m_i} = \frac{m_1 z_1 + m_2 z_2 + \cdots + m_n z_n}{m_1 + m_2 + \cdots + m_n}.$$

From this definition, it is seen that the center of gravity (or the center of mass) is the centroid of the points x_i, y_i, and z_i for the weighting factors m_i. Its position *with respect to the particles* is therefore (Sec. 22) independent of the coordinate system chosen. It is recommended that this be verified directly by a translation and a rotation of axes according to the usual geometrical methods.

77. The Center of Gravity of a Continuous Mass.—If the system of particles forms a continuous mass, the sums which enter into the expressions for the coordinates of the center of gravity

pass over into definite integrals, which are commonly thought of as the limit of a sum. Each mass m_i decreases as n increases but in such a way that this sum remains finite. On writing dm instead of m_i, dropping the subscripts which are not needed in the calculus notation, and replacing the summation symbols by the integration signs, there results

$$\bar{x} = \frac{\int_B x dm}{\int_B dm}, \quad \bar{y} = \frac{\int_B y dm}{\int_B dm}, \quad \text{and } \bar{z} = \frac{\int_B z dm}{\int_B dm},$$

the integration to be extended over the entire body.

Actual bodies, of course, are not regarded as continuous in the mathematical sense; but the continuity ideal furnishes an approximation which is sufficiently exact for most purposes.

78. The Density at a Point.—The average density of a body is its mass divided by its volume (dimensions ML^{-3}),

$$\text{average density} = \frac{\text{mass}}{\text{volume}}.$$

The density at a point is the limit for diminishing volumes of the average density of volumes which contain that point. Let it be supposed that the density of a body varies in a continuous manner and that at the point x, y, z its value is the value of the function $\sigma(x, y, z)$ at that point. Then, in rectangular coordinates,

$$dm = \sigma dxdydz.$$

If it is desired to use the polar coordinates

$$x = r \cos \varphi \cos \theta,$$
$$y = r \cos \varphi \sin \theta,$$
$$z = r \sin \varphi,$$

the expression for the element of mass is

$$dm = \sigma r^2 \cos \varphi d\varphi d\theta dr.$$

79. The Center of Gravity of an Area or of a Line.—In case the body is a thin plane sheet of uniform density, like a sheet of paper, it is necessary to determine only two of the coordinates of the center of gravity, and *the center of gravity of an area* is spoken of. In case the body is a thin rod, even though the density be not uniform, there is but one coordinate to be determined, and *the center of gravity of a line* is spoken of. The center of gravity of a uniform straight rod is evidently the center of the rod.

For a thin plane sheet, not necessarily uniform, the density is a function of two coordinates $\sigma(x, y)$ and

$$dm = \sigma dx dy,$$

or, in polar coordinates,

$$x = r \cos \theta, \qquad y = r \sin \theta, \qquad dm = \sigma(r, \theta) r dr d\theta.$$

For a thin rod $\qquad dm = \sigma(x) dx.$

80. Center of Gravity of Portions of a Body.—It happens at times that a body can be thought of as the union of two or more masses for each of which the center of mass is easily found. Let M_1 and M_2 be the masses of two parts of a body of mass M, so that

$$M = M_1 + M_2;$$

let $\bar{x}_1$, $\bar{y}_1$, $\bar{z}_1$, and $\bar{x}_2$, $\bar{y}_2$, $\bar{z}_2$ be their respective centers of gravity. If $\bar{x}$, $\bar{y}$, $\bar{z}$ is the center of gravity of the entire mass M, then

$$\bar{x} = \frac{M_1\bar{x}_1 + M_2\bar{x}_2}{M_1 + M_2}, \qquad \bar{y} = \frac{M_1\bar{y}_1 + M_2\bar{y}_2}{M_1 + M_2}, \qquad \bar{z} = \frac{M_1\bar{z}_1 + M_2\bar{z}_2}{M_1 + M_2}.$$

Let the particles of the first portion be $m_1, \cdots, m_k$ and of the second portion $m_{k+1}, \cdots, m_n$. Then, by definition,

$$\bar{x}_1 = \frac{m_1x_1 + m_2x_2 + \cdots + m_kx_k}{m_1 + m_2 + \cdots + m_k}, \qquad \bar{x}_2 = \frac{m_{k+1}x_{k+1} + \cdots + m_nx_n}{m_{k+1} + \cdots + m_n}.$$

But

$$\begin{aligned}\bar{x} &= \frac{m_1x_1 + m_2x_2 + \cdots + m_kx_k + m_{k+1}x_{k+1} + \cdots + m_nx_n}{m_1 + m_2 + \cdots + m_k + m_{k+1} + \cdots + m_n}, \\ &= \frac{(m_1x_1 + m_2x_2 + \cdots + m_kx_k) + (m_{k+1}x_{k+1} + \cdots + m_nx_n)}{(m_1 + m_2 + \cdots + m_n) + (m_{k+1} + \cdots + m_n)}, \\ &= \frac{(m_1 + m_2 + \cdots + m_k)\bar{x}_1 + (m_{k+1} + \cdots + m_n)\bar{x}_2}{(m_1 + m_2 + \cdots + m_k) + (m_{k+1} + \cdots + m_n)},\end{aligned}$$

or

$$\bar{x} = \frac{M_1\bar{x}_1 + M_2\bar{x}_2}{M_1 + M_2};$$

and similarly for $\bar{y}$ and $\bar{z}$. From this it follows that the center of gravity of a body is unaltered if the body is divided up into parts, and the mass of each part is concentrated at its center of gravity.

81. Symmetry.—If a body is symmetrical with respect to a certain plane, the center of gravity will lie in the plane of symmetry, but the symmetry must hold with respect to the density as well as to the geometrical figure. If a body has two planes of symmetry, the center of gravity will lie on the line in which the planes of symmetry intersect. If there are three planes of

symmetry which intersect in a point, the center of gravity will lie at the point of intersection of the three planes.

For example, the center of gravity of a homogeneous body (*i.e.*, constant density), which has a figure of revolution, will lie on the axis of revolution. The center of gravity of homogeneous parallelepipeds, spheroids, and ellipsoids coincides with the center of figure.

82. Example of a Line.—Let the density at any point of a thin rod of length l be proportional to the distance of the point from one end of the line. It is required to find the center of gravity of the line.

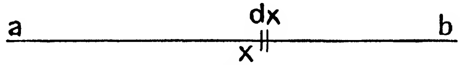


Fig. 27.

Let $\overline{ab}$ be the rod, dx an element of the rod at the distance x from the end a, and let kx be the density of the element dx. Then the element of mass dm at dx is

$$dm = kxdx.$$

Then

$$\int_B xdm = k\int_0^l x^2dx = \frac{1}{3}kl^3;$$

and

$$\int_B dm = k\int_0^l xdx = \frac{1}{2}kl^2.$$

Therefore,

$$\bar{x} = \frac{\int xdm}{\int dm} = \frac{\frac{1}{3}kl^3}{\frac{1}{2}kl^2} = \frac{2}{3}l. \qquad (1)$$

It will be observed that the center of gravity does not depend upon the factor of proportionality k.

83. The Center of Gravity of a Triangle.—Let the triangle abc be divided up into infinitesimal strips parallel to the base $\overline{bc}$. The length, and therefore the mass, of each strip is proportional to its distance x from the vertex of the triangle. The center of gravity of each strip is at the center of the strip, and therefore lies on the line $\overline{ad}$ which joins the vertex to the center of the base. Imagine the mass of each strip concentrated at its center of gravity, and therefore on the line $\overline{ad}$. The density along the line $\overline{ad}$ is now proportional to the distance from a. Hence, by Eq.

(82.1), the center of gravity is at the point which is two-thirds of the distance from a to d.

It can be regarded in another way. Since the center of gravity of each strip lies on the median $\overline{ad}$ which bisects the base the center of gravity of the triangle also lies on $\overline{ad}$. But either one of the other two sides could have been regarded as the base and, therefore, the center of gravity lies on all three of the medians $\overline{ad}$, $\overline{be}$, and $\overline{cf}$. It is known from geometry that the three medians of a triangle meet at a point which is two-thirds of the distance from the vertex to the center of the opposite side.

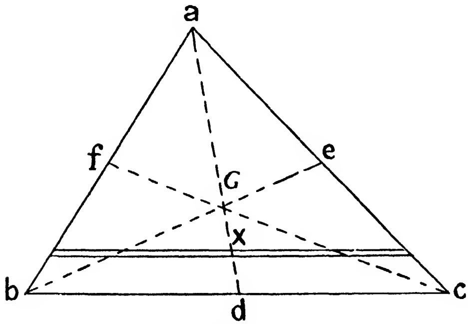


FIG. 28.

The center of gravity of a polygon of any shape can be determined by breaking up the polygon into triangles.

84. Pyramids and Cones.—Let a-$bcdef$ be a homogeneous pyramid, and let g be the center of gravity of the base. Divide the pyramid into thin laminae of equal thickness by planes parallel to the base. The line $\overline{ag}$ passes through the center of gravity of each of them. The mass of each lamina is proportional to the square of its distance from a measured along ag. If the mass of each lamina be concentrated at its center of gravity, the density along the line $\overline{ag}$ will be proportional to the square of the distance from a. But for such a line

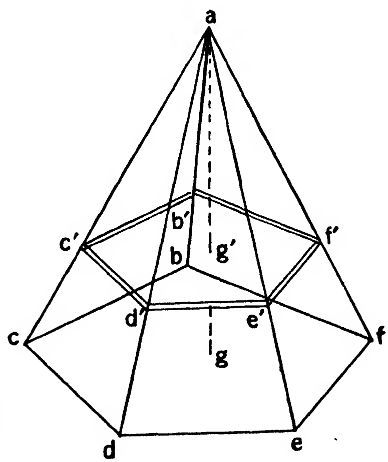


FIG. 29.

$$\bar{x} = \frac{k\int_0^l x^3dx}{k\int_0^l x^2dx} = \frac{3}{4}l;$$

hence, the center of gravity of a pyramid lies on the line joining the vertex to the center of gravity of the base and is one-fourth of the distance from the base to the vertex.

Since this result is independent of the number of sides of the base, the base can have any number of sides, or be bounded even by a plane curve, in which case the pyramid becomes a cone.

85. An Arc and a Sector of a Circle.—It is evident from symmetry that the center of gravity of a circular arc lies on the radius which bisects the arc. Let this bisector be taken as the x-axis

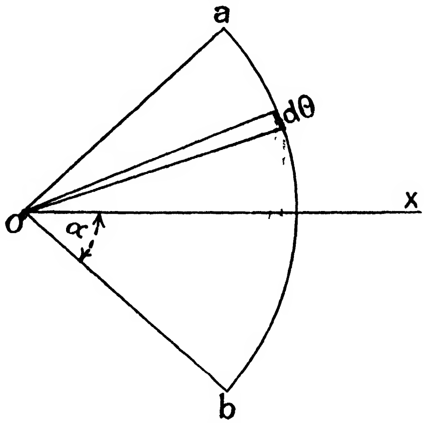


FIG. 30.

with the origin at the center of the circle. Then $\bar{y}$ is zero if the density of the arc σ is uniform as is supposed to be the case.

Using polar coordinates,

$$dm = \sigma r d\theta$$

and

$$\bar{x} = \frac{\sigma\int_{-\alpha}^{+\alpha} r\cos\theta \cdot r d\theta}{\sigma\int_{-\alpha}^{+\alpha} r d\theta} = r\cdot\frac{\sin\alpha}{\alpha}.$$

If the sector Oab is divided up into narrow strips by concentric circular arcs, it is seen from this result and the analogy with a triangle that the center of gravity of the sector Oab is

$$\bar{x} = \frac{2}{3}r\frac{\sin\alpha}{\alpha}, \qquad \bar{y} = 0.$$

86. A Homogeneous Hemisphere.—Let the hemisphere be the upper half of the sphere

$$x^2 + y^2 + z^2 = a^2.$$

By virtue of symmetry, the center of gravity lies on the z-axis. Let the hemisphere be divided into thin laminae of thickness dz by planes parallel to the xy-plane. The area of the lamina bounded by planes whose distances from the xy-plane are z and $z + dz$ is $\pi(a^2 - z^2)$ and its mass is $\pi\sigma(a^2 - z^2)dz$. Imagine this mass concentrated at the center of gravity of the lamina. If this is done for every lamina, the mass will all lie on the z-axis, the law of density being $\pi\sigma(a^2-z^2)$. Therefore,

$$\bar{z} = \frac{\pi\sigma\int_0^a (a^2 - z^2)z\,dz}{\pi\sigma\int_0^a (a^2 - z^2)dz} = \frac{3}{8}a.$$

87. Any Homogeneous Solid of Revolution.—Let the z-axis be the axis of revolution. It follows from symmetry that

$$\bar{x} = \bar{y} = 0.$$

Let

$$x = f(z)$$

be the equation of the intersection of the xz-plane with the bounding surface. The area of a cross-section of the solid perpendicular to its axis at a distance z from the origin is $\pi x^2 = \pi f^2(z)$. Hence, if the mass be concentrated on the z-axis, which passes through the center of gravity of every cross-section, the center of gravity of a straight line the density of which is $f^2(z)$ must be computed.

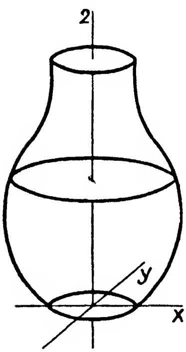


FIG. 31.

Hence,

$$\bar{z} = \frac{\int_a^b z f^2(z)dz}{\int_a^b f^2(z)dz}.$$

88. First Theorem of Pappus.—The area of a surface generated by revolving a plane curve about any axis in its plane is equal to the length of the curve multiplied by the length of the circumference described by its center of gravity.

Let the axis about which the curve is revolved be the z-axis, and let ds be an arc element of the curve. If the density of the curve is taken equal to unity, $dm = ds$. Then the x-coordinate of the center of gravity is

$$\bar{x} = \frac{\int x\,ds}{\int ds},$$

and therefore

$$2\pi\bar{x}s = 2\pi\int xds.$$

But $2\pi\bar{x}$ is the length of the circumference described by the center of gravity, and $2\pi\int xds$ is the area of the surface generated by the curve. The theorem is therefore proved.

89. Second Theorem of Pappus.—The volume generated by any plane area revolving about any axis in its plane, which does not penetrate the area, is equal to the generating area multiplied by the length of the circumference described by its center of gravity.

Let the generating area lie in the xz-plane, and the z-axis be the axis of revolution. If the density of the area be taken equal to unity, $dm = dxdz$, and the x-coordinate of the center of gravity of the generating area is

$$\bar{x} = \frac{\int\int xdxdz}{\int\int dxdz}.$$

Therefore,

$$2\pi\bar{x}A = \pi\int x^2dz,$$

the integral being taken around the curve

$$x = f(z)$$

which bounds the area. But this integral is the volume generated by the area,[1] and $2\pi\bar{x}$ is the circumference described by the center of gravity of the area A.

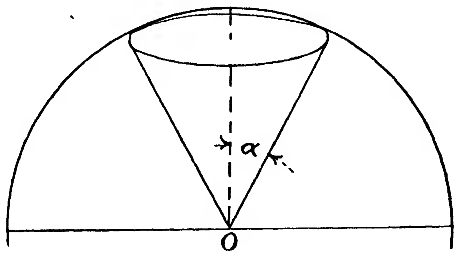


FIG. 32.

90. Example with Polar Coordinates.—To find the center of gravity of a homogeneous solid which is bounded by a cone and a sphere whose center is at the apex of the cone. Let a be the radius of the sphere, α the generating angle of the cone and the z-axis the axis of the cone. Since the z-axis is an axis of sym-

[1] See WILLIAMSON'S, "Integral Calculus," p. 254.

metry the center of gravity lies on it. It is necessary therefore to compute only the z-coordinate, which is defined by

$$\bar{z} = \frac{\int_B z\,dm}{\int_B dm}.$$

Using polar coordinates this expression becomes

$$\bar{z} = \frac{\displaystyle\int_{\frac{\pi}{2}-\alpha}^{\frac{\pi}{2}} \int_0^a \int_0^{2\pi} (r \sin \varphi) r^2 \cos \varphi \, d\varphi dr d\theta}{\displaystyle\int_{\frac{\pi}{2}-\alpha}^{\frac{\pi}{2}} \int_0^a \int_0^{2\pi} r^2 \cos \varphi \, d\varphi dr d\theta}$$

$$= \frac{\displaystyle\int_{\frac{\pi}{2}-\alpha}^{\frac{\pi}{2}} \int_0^a r^3 \sin \varphi \cos \varphi \, d\varphi dr}{\displaystyle\int_{\frac{\pi}{2}-\alpha}^{\frac{\pi}{2}} \int_0^a r^2 \cos \varphi \, d\varphi dr} = \frac{3}{4} a \frac{\displaystyle\int_{\frac{\pi}{2}-\alpha}^{\frac{\pi}{2}} \sin \varphi \cos \varphi \, d\varphi}{\displaystyle\int_{\frac{\pi}{2}-\alpha}^{\frac{\pi}{2}} \cos \varphi \, d\varphi}$$

Hence, finally

$$\bar{z} = \frac{3}{16} a \frac{1 - \cos 2\alpha}{1 - \cos \alpha} = \frac{3}{8} a (1 + \cos \alpha).$$

Problems V

Verify the following statements:

1. The distance of the center of gravity of a sector of angle 2θ from the center of a circle of radius a is

$$\bar{x} = \frac{2}{3} a \frac{\sin \theta}{\theta}.$$

2. The distance of the center of gravity of a segment of angle 2θ from the center of a circle of radius a is

$$\bar{x} = \frac{2}{3} a \frac{\sin^3 \theta}{\theta - \sin \theta \cos \theta}.$$

3. The distance of the center of gravity of the volume of uniform density contained between a hemisphere and a right circular cone of the same base and altitude from the base is one-half this common altitude.

4. If the vertex of a right circular cone lies in the surface of a sphere, if its generating angle is α, if its axis coincides with a diameter, and if its base is the intercepted portion of the sphere, the distance of its center of gravity from the vertex is

$$\bar{x} = \frac{1 - \cos^6 \alpha}{1 - \cos^4 \alpha} a.$$

5. The center of gravity of three equal particles at the three vertices of a triangle coincides with the center of gravity of the area of the triangle.

6. The center of gravity of three particles at the three vertices of a triangle, the masses of which are proportional to the lengths of the opposite sides, is at the center of the inscribed circle.

7. The center of gravity of a spherical cap cut off from a sphere of radius a by a plane at a distance b from the center of the sphere is

$$\frac{1}{4}\frac{(a-b)(3a+b)}{(2a+b)} \text{ above the base of the cap,}$$

or

$$\frac{3}{4}\frac{(a+b)^2}{(2a+b)} \text{ from the center of the sphere.}$$

8. The distance from the base to the center of gravity of a solid hemisphere whose density varies as the square of the distance from the center of the base is five-twelfths of the radius.

9. The coordinates of the center of gravity of an octant of an ellipsoid are

$$\bar{x} = \frac{3}{8}a, \qquad \bar{y} = \frac{3}{8}b, \qquad \bar{z} = \frac{3}{8}c.$$

10. The area of the surface of an anchor ring which is generated by revolving a circle of radius a about an axis in its own plane and at a distance $b > a$ from the center of the circle is $4\pi^2ab$.

11. The volume of the tore generated by revolving an ellipse of semiaxes a and b about any axis in its plane at a distance c from the center of the ellipse (provided the axis does not penetrate the ellipse) is $2\pi^2abc$.

12. The area of the surface and the volume of the solid generated by revolving a single wave of a cycloid about its base are respectively twice and five times as great as the surface and volume generated by revolving it about the tangent at its highest point.

13. If an equilateral triangle of side a is revolved about one of its sides as an axis, the area of the surface generated is $\sqrt{3}\pi a^2$ and its volume is $\pi a^3/4$.

II. MOMENTS OF INERTIA

91. The Moment of Inertia Defined.—The second of the geometric concepts referred to in Sec. 75 is *the moment of inertia* which is a quadratic expression of the coordinates, the center of gravity being a linear expression. It plays a rôle in the theory of rotating bodies which is analogous to that which mass plays in the theory of translation. As it is not an important concept for that portion of the theory of mechanics which is considered in this volume, the subject will be considered here but briefly.

Suppose there are n particles of mass $m_1, m_2, \cdots, m_n$, with coordinates $x_1, y_1, z_1; x_2, y_2, z_2; \cdots; x_n, y_n, z_n$. *The moment of inertia I of this system of particles with respect to a plane, line, or point is the sum of the products of the mass of the*

particles into the square of the perpendicular distance to the plane or line (or merely the square of the distance in the case of a point).

$$I = \sum m_i p_i^2 = m_1 p_1^2 + m_2 p_2^2 + \cdots + m_n p_n^2,$$

where p_i is the perpendicular distance mentioned.

If the system of particles forms a continuous body, this sum passes over into the definite integral

$$I = \int_B p^2 dm,$$

the integral being taken over the entire body.

92. The Various Types of Moment of Inertia.—If I_{yz} is the moment of inertia with respect to the yz-plane, then

$$I_{yz} = \sum m_i x_i^2, \qquad \text{or} \qquad \int_B x^2 dm;$$

and similarly,

$$I_{zx} = \sum m_i y_i^2, \qquad \text{or} \qquad \int_B y^2 dm;$$

$$I_{xy} = \sum m_i z_i^2, \qquad \text{or} \qquad \int_B z^2 dm.$$

If I_x, I_y, and I_z are the moments of inertia with respect to the x-, y-, z-axes, repectively, then

$$I_x = \sum m_i(y_i^2 + z_i^2), \qquad \text{or} \qquad \int_B (y^2 + z^2)dm = I_{zx} + I_{xy},$$

$$I_y = \sum m_i(z_i^2 + x_i^2), \qquad \text{or} \qquad \int_B (z^2 + x^2)dm = I_{xy} + I_{yz},$$

$$I_z = \sum m_i(x_i^2 + y_i^2), \qquad \text{or} \qquad \int_B (x^2 + y^2)dm = I_{yz} + I_{zx}.$$

The moment of inertia with respect to the origin is

$$\begin{aligned} I_0 &= \sum m_i(x_i^2 + y_i^2 + z_i^2), \qquad \text{or} \qquad \int_B (x^2 + y^2 + z^2)dm, \\ &= I_{xy} + I_{yz} + I_{zx}, \\ &= \frac{1}{2}(I_x + I_y + I_z). \end{aligned}$$

Since the choice of the coordinate system is quite arbitrary, it is seen from these relations that

The moment of inertia with respect to any line is the sum of the moments of inertia with respect to any two mutually perpendicular planes which pass through that line.

The moment of inertia with respect to any point is equal to the sum of the moments of inertia with respect to any three mutually perpendicular planes which pass through that point, or one-half

the sum of the moments of inertia with respect to any three mutually perpendicular lines through that point.

93. Products of Inertia.—Similar expressions which involve the cross-products of the coordinates are called *products of inertia*, namely,

$$\sum m_i x_i y_i, \qquad \sum m_i y_i z_i, \qquad \sum m_i z_i x_i,$$

or

$$\int_B xy\,dm, \qquad \int_B yz\,dm, \qquad \int_B zx\,dm.$$

94. The Radius of Gyration.—Suppose that the entire mass M of the body is concentrated into a single particle of mass M at a distance k from a given axis with respect to which the moment of inertia of the body is I. If k is chosen so that

$$Mk^2 = I,$$

it is called the *radius of gyration* of the body with respect to that axis.

The *principal radius of gyration* is the radius of gyration with respect to a parallel axis which passes through the center of gravity of the body or system of particles.

95. The Principal Radius of Gyration is a Minimum.—Let I be the moment of inertia with respect to a given axis, which will be taken as the z-axis of the coordinate system. Let k be the radius of gyration of the body with respect to this axis, and let k_0 be the radius of gyration of the body with respect to a parallel axis through the center of gravity of the body, so that k_0 is a principal radius of gyration. Let x_0, y_0 and z_0 be the coordinates of the center of gravity and

$$x = x_0 + \xi, \qquad y = y_0 + \eta, \qquad z = z_0 + \zeta,$$

so that, ξ, η, and ζ are the coordinates of a particle with respect to the center of gravity. Then,

$$Mk_0^2 = \sum m_i(\xi_i^2 + \eta_i^2)$$

and

$$\sum m_i\xi_i = \sum m_i\eta_i = \sum m_i\zeta_i = 0.$$

Also

$$\begin{aligned} Mk^2 &= \sum m_i(x_i^2 + y_i^2), \\ &= \sum m_i[(x_0 + \xi_i)^2 + (y_0 + \eta_i)^2], \\ &= \sum m_i(\xi_i^2 + \eta_i^2) + M(x_0^2 + y_0^2) + 2x_0\sum m_i\xi_i + 2y_0\sum m_i\eta_i. \end{aligned}$$

If p is the perpendicular distance between the two axes, then

$$p^2 = x_0^2 + y_0^2;$$

and the above expression reduces to

$$Mk^2 = Mk_0^2 + Mp^2,$$

or

$$k^2 = k_0^2 + p^2.$$

Thus k, k_0 and p can be represented by the three sides of a right triangle, as is shown in Fig. 33. From this relation it is seen that, for any system of parallel axes, the radius of gyration is least for that one which passes through the center of gravity; that is, the principal radius of gyration is a minimum one.

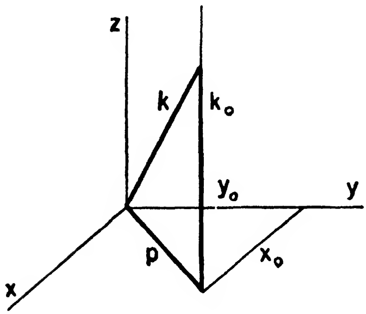


Fig. 33.

96. The Moment of Inertia of an Area.—If the body under discussion is a thin plane lamina, the element of mass dm is equal to the mass per unit area σ times the element of area da. In this case, if σ is constant, the moment of inertia is

$$I = \sigma \int_A r^2 da$$

where A is the total area, and the integral

$$\int_A r^2 da$$

is called the *moment of inertia of the area.*

In the same way there arises the term *moment of inertia of a volume*

$$\int_V r^2 dv$$

and the *moment of inertia of a line*

$$\int_L r^2 dl.$$

97. The Moment of Inertia of a Rod.—What is the moment of inertia of a uniform rod of length l with respect to an axis perpendicular to the rod through its center?

Let σ be the mass of the rod per unit length and $M = l\sigma$ its total mass. Then

$$I = \sigma \int_{-\frac{l}{2}}^{+\frac{l}{2}} x^2 dx = \frac{1}{12} Ml^2;$$

and its radius of gyration is

$$k = \frac{l}{2\sqrt{3}}.$$

With respect to a parallel axis at a distance d from the center,

$$I = M\left(\frac{1}{12}l^2 + d^2\right),$$

so that, if the axis is at the end of the rod,

$$I = \frac{1}{3}Ml^2.$$

98. Moment of Inertia of a Parallelepiped.—Let the parallelepiped be homogeneous with edges a, b, and c and let the axis, with respect to which the moment of inertia is taken, pass through its center parallel to the edge c. Then

$$\begin{aligned} I &= \sigma\int_{-\frac{a}{2}}^{+\frac{a}{2}}\int_{-\frac{b}{2}}^{+\frac{b}{2}}\int_{-\frac{c}{2}}^{+\frac{c}{2}}(x^2 + y^2)dxdydz \\ &= \sigma c\int_{-\frac{a}{2}}^{+\frac{a}{2}}\int_{-\frac{b}{2}}^{+\frac{b}{2}}(x^2 + y^2)dxdy \\ &= \frac{1}{12}\sigma abc(a^2 + b^2) = \frac{1}{12}M(a^2 + b^2). \end{aligned}$$

99. Moment of Inertia of a Sphere.—Let R be the radius of a homogeneous sphere. The moment of inertia with respect to a diameter, the z-axis say, is

$$\begin{aligned} I &= \sigma\int_0^R\int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}}\int_0^{2\pi}\rho^4\cos^3\varphi\, d\rho d\varphi d\theta, \\ &= \frac{8}{15}\pi\sigma R^5, \\ &= \frac{2}{5}MR^2. \end{aligned}$$

Problems VI

Verify the following statements:

1. *Routh's Rule.*—The square of the radius of gyration of a homogeneous (*a*) rod, rectangular lamina, or parallelepiped; (*b*) elliptical or circular disk; (*c*) ellipsoid, spheroid or sphere about an axis of symmetry is

$$k^2 = \frac{\text{sum of the squares of the perpendicular semiaxes}}{3,\ 4,\ \text{or}\ 5}$$

according as the object belongs to class (*a*), (*b*), or (*c*).

2. The radius of gyration of a triangle with respect to an axis through one vertex and parallel to the opposite side is the altitude of the triangle divided by the square root of 2.

3. If b is the base of a triangle and l is the distance from the vertex to the midpoint of the base, the square of the radius of gyration of the triangle with respect to an axis through the vertex and perpendicular to the plane of the triangle is

$$k^2 = \frac{1}{2}l^2 + \frac{1}{24}b^2.$$

4. With respect to the diagonal of a rectangle of which the sides are a and b,

$$k^2 = \frac{a^2b^2}{6(a^2 + b^2)}.$$

5. The radius of gyration of a cube about a diagonal of which the length is d is

$$k = \frac{\sqrt{5}d}{6}.$$

6. For a regular tetrahedron with edges of length l, about an axis through a vertex and perpendicular to the opposite face,

$$k = \frac{l}{\sqrt{20}}.$$

7. The radius of gyration of a spherical shell of radius r about a diameter is

$$k = \sqrt{\frac{2}{3}}r.$$

8. The radius of gyration of the arc of one complete wave of cycloid of length a is

$$k = \frac{a}{\sqrt{30}}.$$

9. The moment of inertia of a lamina in the form of a regular polygon with n sides about a line through its center and perpendicular to its plane, in terms of the perpendicular distance p from the center to the sides is

$$I = \frac{1}{6}Mp^2\left(3 + \tan^2\frac{\pi}{n}\right).$$

10. The moment of inertia of a homogeneous cone about an axis through its center of gravity and perpendicular to its axis of figure is

$$I = \frac{3}{80}Mh^2(1 + 4\tan^2\alpha).$$

11. The radius of gyration of an elliptical lamina about its major axis is

$$k = \frac{1}{2}a.$$

12. The moment of inertia of an anchor ring of which the radius of the generating circle is a and the distance from the axis of revolution to the center of the generating circle is $b > a$, about the axis of revolution, is

$$I = M\left(b^2 + \frac{3}{4}a^2\right).$$

PART II

STATICS

CHAPTER VII

THE STATICS OF A PARTICLE

100. Definitions.—By the term "statics" is meant the study of the conditions under which a particle or a body remains at rest. A particle or a body which remains at rest under a given set of forces is said to be in *equilibrium* under those forces.

101. Theorem I.—*If a particle is in equilibrium under the action of a single force* $\mathbf{F}$, *then* $\mathbf{F}$ *is equal to zero.*

According to Newton's second law the rate of change of momentum is proportional to the force acting. If the particle remains at rest its momentum is constantly zero. Its rate of change of momentum is zero, and therefore $\mathbf{F}$ is zero.

102. Theorem II.—*If a particle is in equilibrium under the action of two forces only,* $\mathbf{F}_1$ *and* $\mathbf{F}_2$, *then*

$$\mathbf{F}_1 + \mathbf{F}_2 = 0,$$

and the two forces are equal in magnitude and are oppositely directed.

FIG. 34.

Since forces are vectors, $\mathbf{F}_1$ and $\mathbf{F}_2$, acting at the same point, are equivalent to a single force $\mathbf{F}$; that is,

$$\mathbf{F}_1 + \mathbf{F}_2 = \mathbf{F}.$$

Since the particle is in equilibrium under the action of a single force $\mathbf{F}$, by theorem I, $\mathbf{F}$ is zero and therefore

$$\mathbf{F}_1 + \mathbf{F}_2 = 0;$$

that is, the forces are equal in magnitude and oppositely directed.

103. Theorem III.—*If a particle is in equilibrium action of three forces* $\mathbf{F}_1$, $\mathbf{F}_2$, *and* $\mathbf{F}_3$, *then*

$$\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 = 0.$$

The proof is omitted as it is quite similar to the proof of theorem II. As has already been seen in Chap. I, problem 2, the condition

$$\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 = 0$$

implies that if the three vectors are taken sequentially, so that the origin of the second coincides with the terminus of the first, and so on, the three vectors will form a closed triangle.

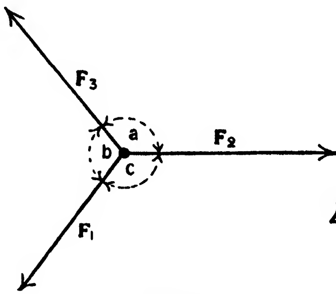


Fig. 35.

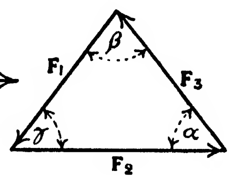


Fig. 35a.

g. 35 represent the three forces acting upon the particle, he angles between them be represented by a, b, and c. . 35a be the same vectors when arranged sequentially, and let the corresponding angles be α, β, and γ. Then

$$a = \pi - \alpha, \qquad b = \pi - \beta, \qquad c = \pi - \gamma.$$

Since F_1, F_2, and F_3 are the three sides of a triangle, it follows from the sine law of trigonometry that

$$\frac{F_1}{\sin \alpha} = \frac{F_2}{\sin \beta} = \frac{F_3}{\sin \gamma};$$

and therefore

$$\frac{F_1}{\sin a} = \frac{F_2}{\sin b} = \frac{F_3}{\sin c}.$$

This formula suggests a very useful form of theorem III known by the name of Lami's theorem.

Lami's Theorem.—If a particle is in equilibrium under the action of three forces, the three forces lie in the same plane and the magnitude of each force is proportional to the sine of the angle between the other two.

104. Theorem IV.—*If a particle is in equilibrium under the action of n forces* $\mathbf{F}_1, \mathbf{F}_2, \cdots, \mathbf{F}_n$, *then*

$$\mathbf{F}_1 + \mathbf{F}_2 + \cdots + \mathbf{F}_n = 0,$$

and the vectors representing these forces, if arranged sequentially, will form a closed polygon.

The n forces acting simultaneously on a single particle are equivalent to a single force

$$\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 + \cdots + \mathbf{F}_n,$$

and, since the particle is in equilibrium,

$$\mathbf{F} = 0,$$

by theorem I. Hence, the resultant of the n vectors is zero and, if the vectors are placed end to end, the terminus of the last vector will coincide with the origin of the first, and the vector polygon will be closed. The vectors need not all lie in the same plane; so that, in general, the vector polygon is not a plane polygon.

105. Components of Forces Along Three Rectangular Axes.—Let **i**, **j**, and **k** be three mutually perpendicular unit vectors (Sec. 21). Each force can be expressed in terms of these three vectors; thus

$$\mathbf{F}_s = x_s\mathbf{i} + y_s\mathbf{j} + z_s\mathbf{k}.$$

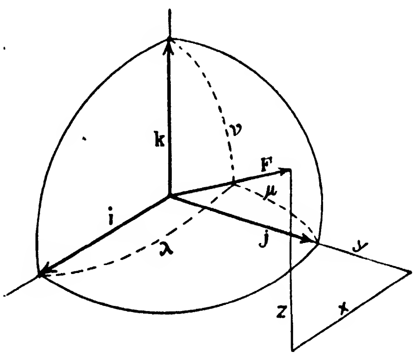


FIG. 36.

On taking the sum of all of the forces, there results

$$\sum \mathbf{F}_s = (x_1 + x_2 + \cdots + x_n)\mathbf{i} + (y_1 + y_2 + \cdots + y_n)\mathbf{j} + (z_1 + z_2 + \cdots + z_n)\mathbf{k} = 0;$$

from which it follows that

$$\begin{aligned} x_1 + x_2 + \cdots + x_n &= 0, \\ y_1 + y_2 + \cdots + y_n &= 0, \\ z_1 + z_2 + \cdots + z_n &= 0; \end{aligned}$$

and the sum of the projections upon any axis whatever is zero.

If λ_s, μ_s, and ν_s are the direction angles of $\mathbf{F}_s$ with respect to the axes of **i**, **j**, and **k**, then

$$x_s = F_s \cos \lambda_s, \qquad y_s = F_s \cos \mu_s, \qquad z_s = F_s \cos \nu_s;$$

and, since

$$\cos^2 \lambda_s + \cos^2 \mu_s + \cos^2 \nu_s = 1,$$

which is true for the direction cosines of any line, it follows that

$$x_s^2 + y_s^2 + z_s^2 = F_s^2.$$

106. Common Types of Force.—The theorems which have just been proved are independent of the physical nature of the forces which are acting, but as everyone knows there is a great variety of forces. A few of the commonest will be discussed briefly, namely, weight, tension in strings and chains, the reaction of surfaces, and friction.

107. Weight.—Every particle is acted upon by the attraction of the earth in accordance with Newton's law of gravitation, and the magnitude of the attraction is the weight of the particle. Its direction is downward; in fact, the meaning of *down* is *in the direction of the acceleration of the earth.* If a mass is suspended by a string and is at rest, the mass is acted upon by two forces, namely, its weight **W** and the tension of the string **T**. These two forces are equal in magnitude and lie in the same straight line, which is called the *plumb line* or the *vertical.*

Fig. 37.

108. Tension of Strings and Chains.—In most of the instances in which strings and chains are employed, the weight of the string or chain is sufficiently small in comparison with the other forces that it may be neglected altogether. Unless something to the contrary is stated, strings and chains will be regarded as weightless. The term "a light string" is used to call attention to the fact that the weight of the string is disregarded and is contrasted with the term "a heavy string" when the weight of the string is important.

A string or a chain can be regarded as a series of particles which are held together by forces. The nature of these forces does not need to be considered beyond the fact that they act only on adjacent particles and along the straight line which joins their centers. Under these conditions they will be perfectly flexible,

by which is meant that they will take the form of any desired curve without offering any resistance to their being bent or curved.

If a string or chain is in equilibrium under the action of two forces $\mathbf{T}_1$ and $\mathbf{T}_2$ (not zero), acting at the extremities, then each particle of the string separately is in equilibrium under the two forces which attach it to the adjacent particles. Hence, by theorem II, these two forces lie in a straight line and are equal in magnitude. In Fig. 38, the particle a is acted on by $\mathbf{T}_1$ and the force which attaches it to the particle b. Likewise, the force joining b to c is equal to the force joining b to a (which, of course, is equal and opposite to the force joining a to b by Newton's third law) and lies in the same straight line. Hence, the three particles a, b, and c lie in the same straight line. Continuing step by step,

FIG. 38.

it is shown that all of the particles lie in the same straight line and that all of the forces between the adjacent particles are equal in magnitude to T_1. The last particle, however, is acted on by a force $\mathbf{T}_1$ and a second force $\mathbf{T}_2$ and, since it is in equilibrium, it must be true that

$$\mathbf{T}_1 + \mathbf{T}_2 = 0.$$

From these relations is derived the definition: *The tension of a string at any point is the magnitude of the force joining the adjacent particles at that point.* If, therefore, a perfectly flexible weightless string is in equilibrium under the action of two non-vanishing forces at its extremities, then the string forms a straight line and is under the same tension throughout. It is convenient language to say that the string transmits the force unaltered. It will be shown in Sec. 118 that, if the string is bent around a convex surface that is perfectly smooth (that is, without friction), the force is transmitted unaltered as to magnitude.

109. Elastic and Inelastic Strings.—There are two types of strings: the inelastic and the elastic. The length of an inelastic string is not altered by tension, and all chains are inelastic. This, of course, is merely a convenient fiction. All strings and chains will stretch under tension but, in case the stretch is so small as to be negligible, the chain or string is regarded as inelastic. Elastic strings alter their length perceptibly under tension. It is found

by experiment that, if the tension is not too great, Hooke's law is very closely satisfied.

Hooke's Law.—The increase in length of an elastic string is proportional to its tension. The factor of proportionality, naturally, depends upon the particular string which is used. The *natural length* of a string is its length when its tension is zero, that is, its unstretched length. The *modulus of elasticity* of a string is the force required to double its natural length.

If the natural length of a string is l and its stretched length is x, the amount of the stretch is $x - l$ and, since the tension is proportional to the stretch,

$$T = k(x - l),$$

where k is the factor of proportionality. If λ is its modulus of elasticity, then

$$\lambda = k(2l - l) = kl,$$

so that

$$k = \frac{\lambda}{l},$$

and therefore

$$T = \lambda \frac{x - l}{l}.$$

By means of this formula the value of the modulus of elasticity of a given string can be determined by experiments with small stretches. Many strings, indeed, would break before stretching to double their natural length.

If λ is very great the string is virtually inelastic. It will be observed that λ has the same dimensions as a tension or force.

110. Weight Suspended by Two Inelastic Strings.—A weight of ten pounds is suspended by two inelastic strings of lengths three and four feet, respectively. The strings are attached to two pegs in a horizontal line five feet apart. What are the tensions of the strings?

Let A and B be the pegs and W the weight. In the triangle ABW,

$$AB = 5 \text{ feet}, \qquad AW = 3 \text{ feet}, \qquad BW = 4 \text{ feet}.$$

Therefore, the angle at W is 90°. Denote the angle whose tangent is 3/4 by $\alpha = 36° \, 52'$, so that the angle at B is α. Then W is in equilibrium under the action of three forces, $\mathbf{T}_1$, $\mathbf{T}_2$, and $\mathbf{W}$, namely, the tensions of the two strings and its weight of ten pounds. The angle between $\mathbf{T}_2$ and $\mathbf{W}$ is $90° + \alpha$, and the angle

between $\mathbf{T}_1$ and $\mathbf{W}$ is $180° - \alpha$. Hence, by Lami's theorem (Sec. 103),

$$\frac{10}{\sin 90°} = \frac{T_1}{\sin(90° + \alpha)} = \frac{T_2}{\sin(180° - \alpha)},$$

from which is derived

$$T_1 = 10 \cos \alpha \text{ pounds} = 8 \text{ pounds},$$
$$T_2 = 10 \sin \alpha \text{ pounds} = 6 \text{ pounds}.$$

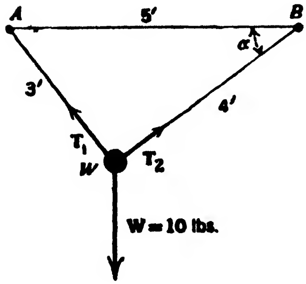


FIG. 39.

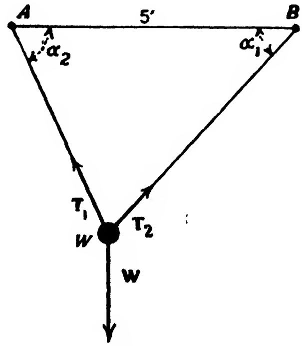


FIG. 40.

111. Weight Suspended by Two Elastic Strings.—In the previous problem, the strings were inelastic. Now suppose that they are elastic with the same modulus of elasticity, λ, equal to twenty pounds, that their natural lengths are three and four feet, that the weight of W is ten pounds, and that the distance between the pegs A and B is five feet.

Since the pegs A and B are fixed, the position of W is determined if the stretched lengths of the strings x_1 and x_2, or the angles opposite them, α_2 and α_1 are known. By Lami's theorem,

$$\frac{T_1}{\sin(90 + \alpha_1)} = \frac{T_2}{\sin(90 + \alpha_2)} = \frac{10}{\sin(\alpha_1 + \alpha_2)};$$

by Hooke's law (Sec. 110),

$$T_1 = 20\left(\frac{x_1 - 3}{3}\right), \qquad T_2 = 20\left(\frac{x_2 - 4}{4}\right);$$

and finally, from the geometry of the triangle ABW,

$$\frac{x_1}{\sin \alpha_1} = \frac{x_2}{\sin \alpha_2} = \frac{5}{\sin(\alpha_1 + \alpha_2)}.$$

These six equations determine the six unknowns α_1, α_2, x_1, x_2, T_1, and T_2. After eliminating x_1, x_2, T_1, and T_2, it is found that

$$\tan \alpha_1 = \frac{\frac{1}{2} + \sin \alpha_2}{\frac{5}{8} - \cos \alpha_2}, \qquad \tan \alpha_2 = \frac{\frac{1}{2} + \sin \alpha_1}{\frac{5}{4} - \cos \alpha_1}.$$

It is not advisable to eliminate any further. If a reasonable guess is made as to the value of α_2 and this value substituted in the first equation, it gives a value of α_1, and this value of α_1 substituted in the second equation gives a new value of α_2. Repeating the process with this new value of α_2, a second value of α_1 is obtained, and so on. The process arrives quickly at the solution

$$\alpha_1 = 48° \, 33', \qquad \alpha_2 = 64° \, 48'.$$

Since

$$T_1 = \frac{10 \cos \alpha_1}{\sin (\alpha_1 + \alpha_2)}, \qquad T_2 = \frac{10 \cos \alpha_2}{\sin (\alpha_1 + \alpha_2)},$$

$$x_1 = \frac{5 \sin \alpha_1}{\sin (\alpha_1 + \alpha_2)}, \qquad x_2 = \frac{5 \sin \alpha_2}{\sin (\alpha_1 + \alpha_2)},$$

it is readily found that

$$T_1 = 7.21 \text{ pounds}, \qquad x_1 = 4.08 \text{ feet},$$
$$T_2 = 4.64 \text{ pounds}, \qquad x_2 = 4.93 \text{ feet}.$$

On comparing these results with the corresponding results of the preceding section, it is seen that the angle between $\mathbf{T}_1$ and $\mathbf{T}_2$ has diminished, as has also each of the tensions T_1 and T_2.

Had the angle diminished to zero, the sum $T_1 + T_2$ would, of course, have been equal to ten pounds.

Problems VII

1. A weight of 100 lb. is suspended by two strings, each of which makes an angle of 60° with the vertical. What are the tensions? *Ans.* 100 lb. each.

2. A weight W suspended by a string is pulled aside by a second horizontal string until the first string makes an angle θ with the vertical. What are the tensions of the two strings? *Ans.* $W \sec \theta$ and $W \tan \theta$.

3. A rubber band is placed around three nails which are driven part way into a board at the vertices of an equilateral triangle of which the side is 6 in. The natural length of the band is 12 in. and its modulus is 2 lb. If the tension of the band is everywhere the same, what is the pressure on each nail? *Ans.* $\sqrt{3}$ lbs.

4. A weight of 100 lb. is suspended by two elastic strings of natural lengths 20 and 25 ft., modulus of elasticity 1000 lb., from two pegs in a horizontal line 40 ft. apart. Determine the conditions of equilibrium. How far is the weight below the line of the pegs? *Ans.* 13.91 ft.; $x_1 = 21.82$; $x_2 = 27.04$; $T_1 = 90.95$; $T_2 = 81.71$; $\alpha_1 = 30° \, 57'$; $\alpha_2 = 39° \, 36'$.

5. A weight W is suspended by three inelastic strings each of length l from the vertices of a horizontal equilateral triangle of which the sides are $s\sqrt{3}$. What is the tension of each string? *Ans.* $T = \dfrac{l}{\sqrt{l^2 - s^2}} \cdot \dfrac{W}{3}$.

112. Reaction of a Surface.—Imagine a body at rest upon a horizontal table. The body is maintained in equilibrium by the action of two forces, its weight **W** acting downward and the reaction of the table or the support of the table. Since the body is in equilibrium under the action of only two forces, these two forces lie in the same straight line, are equal in magnitude, and opposite in direction. Hence, the reaction of the table **R** is equal to the weight in magnitude and is directed upward. It will be observed that the table develops sufficient reaction to support the body (provided the body is not too heavy for the table) but nothing more. Actually, the table yields under the weight of the body, the amount of the yielding being proportional to the weight of the body; but, the modulus of elasticity being very large, the amount of the yield is so small that for the present purposes it can be entirely ignored.

113. Friction.—Imagine now that a string is attached to the body of Sec. 112 and that the string is pulled gently with a force **P** which is horizontal. If P is not too great the body still remains at rest, but it is now under the action of three forces, namely, **P**, **W**, and **R**. The weight **W** is, of course, unchanged, but the reaction of the table **R**, since it must satisfy Lami's theorem, is not now normal to the table. It makes a certain angle α with the normal. It can be resolved into two components one of which, **N**, is normal to the table and is called the normal component, and the other, **F**, parallel to the table, is called the frictional component of the reaction. Evidently, since equilibrium exists,

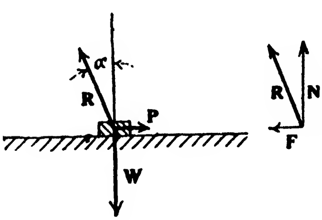


FIG. 41.

$$\mathbf{W} + \mathbf{N} = 0, \qquad \mathbf{P} + \mathbf{F} = 0,$$

and

$$\tan \alpha = \frac{F}{N} = \frac{P}{W}.$$

If the magnitude of the pull P be increased gradually, the angle α will increase until it reaches a certain maximum value, which will be denoted by ϵ and, if the pull be still further increased, the body will start into motion and the equilibrium is destroyed. This maximum angle ϵ which the reaction of the surface makes with the normal to the surface is called *the angle of friction.*

Experiments show that *the angle of friction depends only on the nature of the substances in contact.* Within reasonable limits, it is independent of the areas in contact and of the normal pressures.

Let

$$\tan \epsilon = \mu$$

for the sake of notation; then for any body which is just about to slip upon a given surface

$$F = N\mu,$$

where F is the magnitude of the frictional force parallel to the surface and N is the magnitude of the force normal to the surface. The quantity μ is called *the coefficient of friction.* The following table gives some idea of its values:

COEFFICIENTS OF FRICTION

Wood on wood, dry	0.25–0.50
Metals on metals, dry	0.15–0.20
Metals on metals, wet	0.30
Hemp on oak, dry	0.53
Hemp on oak, wet	0.33

Friction doubtless is due to the irregularities of the surfaces in contact. If the surfaces were smooth in the mathematical sense of the word, there would be no friction. Accordingly, when it is desired to exclude the action of friction it is said that the surfaces are *smooth.* When it is desired to include the action of friction the surfaces are said to be *rough.*

The reaction of a smooth surface is always normal to the surface.

114. Equilibrium on an Inclined Plane.—A body will remain at rest upon an inclined plane if, and only if, the inclination of the plane to the horizontal α is less than, or at most equal to, the angle of friction.

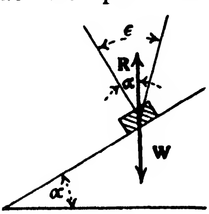


FIG. 42.

Since the body is acted upon only by gravity and by the reaction of the plane, these two forces lie in the same straight line and are equal in magnitude. The reaction **R**, therefore, makes an angle with the normal which also is equal to α. Since slippage can occur only if **R** makes an angle with the normal equal to or greater than the angle of friction, the body will remain at rest only if $\alpha \leqq \epsilon$.

115. Breaking the Equilibrium with a Minimum Force.—A mass of weight W rests upon a rough horizontal plane. What is the magnitude and direction of the smallest force that will just start it into motion?

When the mass is on the point of slipping, the reaction of the surface **R** makes an angle ϵ with the vertical. If the applied force **P** makes an angle θ with the horizontal, Lami's theorem gives

$$\frac{W}{\sin (90° + \epsilon - \theta)} = \frac{P}{\sin (180° - \epsilon)} = \frac{R}{\sin (90° + \theta)},$$

so that

$$P = \frac{W \sin \epsilon}{\cos (\epsilon - \theta)}.$$

Since W and ϵ are given numbers, P is a minimum when $\cos (\epsilon - \theta)$ is a maximum. Hence $\theta = \epsilon$, and the smallest force that will just move the mass is

$$P = W \sin \epsilon,$$

its direction making an angle ϵ with the horizontal.

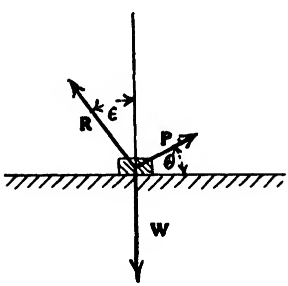


FIG. 43.

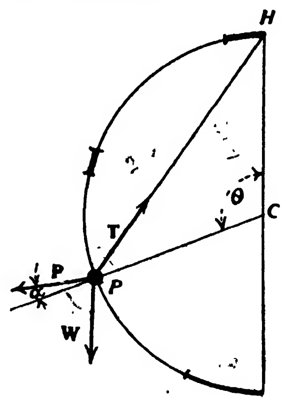


FIG. 44.

116. A Ring on a Vertical Hoop.—A small ring, the weight of which is one ounce, is free to slide on a vertical, circular wire hoop of radius a, the angle of friction being 10°. It is attached to an elastic string which is fastened to the highest point of the hoop. The natural length of the string is the radius of the hoop, and its modulus of elasticity is seven ounces. Find the limiting positions of equilibrium of the ring on the hoop.

The ring is in equilibrium under the action of three forces, *viz.*, its weight **W**, the tension of the string **T**, and the reaction of

the hoop **R**; and, if the ring is on the point of slipping, **R** makes an angle of 10° with the normal to the hoop. In other positions of equilibrium, **R** makes an angle α with the normal which is less than 10°.

Let θ be the angle which the radius to the ring P makes with the radius to the highest point H. If the angle θ is less than 60° the string does not act upon the ring, and, by Sec. 114, the ring is in equilibrium for any value of θ less than 10°. There certainly is no position of equilibrium for $10° < \theta \leqq 60°$. For $\theta > 60°$, it is seen that the angles

$$\widehat{\mathbf{TR}} = \tfrac{1}{2}\pi + \tfrac{1}{2}\theta - \alpha, \widehat{\mathbf{RW}} = \pi - \theta + \alpha, \widehat{\mathbf{WT}} = \tfrac{1}{2}\pi + \tfrac{1}{2}\theta.$$

Then, by Lami's theorem,

$$\frac{R}{\cos \frac{1}{2}\theta} = \frac{T}{\sin (\theta - \alpha)} = \frac{W}{\cos (\frac{1}{2}\theta - \alpha)}.$$

Also, by Hooke's law for the tension of the stretched string,

$$T = 7\left(\frac{2a \sin \frac{1}{2}\theta - a}{a}\right) = 7(2 \sin \tfrac{1}{2}\theta - 1).$$

On equating these two values of T, it is found that, since $W = 1$,

$$\frac{\sin (\theta - \alpha)}{\cos (\frac{1}{2}\theta - \alpha)} = 7(2 \sin \tfrac{1}{2}\theta - 1); \tag{1}$$

and, after solving this equation for α, there results

$$\tan \alpha = \frac{\cos \frac{1}{2}\theta(7 - 12 \sin \frac{1}{2}\theta)}{(1 - 3 \sin \frac{1}{2}\theta)(1 - 4 \sin \frac{1}{2}\theta)}$$

This expression for $\tan \alpha$ changes sign for
$\theta = 2 \sin^{-1} \frac{1}{4} = 29°$ approximately, $\theta = 2 \sin^{-1} \frac{1}{3} = 39°$,

$$\theta = 2 \sin^{-1} \tfrac{7}{12} = 71° \ 22', \text{ and } \theta = \pi.$$

It is positive for $\theta = 60°$, vanishes if $\theta = 2 \sin^{-1} \frac{7}{12} = 71° \ 22'$, and is negative for all values of θ between $71° \ 22'$ and π, which means that the reaction **R** lies on the opposite side of the normal. Lastly, it vanishes for $\theta = \pi$.

In order to find the points at which slippage is about to occur, it is necessary to set $\alpha = \pm 10°$ and to solve for θ. For this purpose, it is better to take Eq. (1) in the form

$$\sin (\theta - \alpha) = \tfrac{7}{6}(\cos (\tfrac{1}{2}\theta - \alpha) - \sin \alpha).$$

For $\alpha = +10°$, there is a single solution, namely, $\theta = 69° \ 9'$. For $\alpha = -10°$, there are two solutions, $\theta = 74° \ 24'$ and $\theta = 155°$

48′. The ring will be in equilibrium, therefore, in the following intervals:

$$0 \leqq \theta < 10°, \quad 69° \, 9' < \theta < 74° \, 24', \quad 155° \, 48' < \theta \leqq 180°.$$

In other intervals of this semicircle, equilibrium cannot exist. There are, of course, symmetrical intervals on the other half of the hoop.

117. Direction of Motion on an Inclined Plane.—A body at rest upon an inclined plane is acted upon by a force in the plane just large enough to start it into motion. In what direction does the body start to move?

Let **W** be the weight of the body, **T** the tension of the string by which it is pulled, and **R** the reaction of the plane. Let α be the

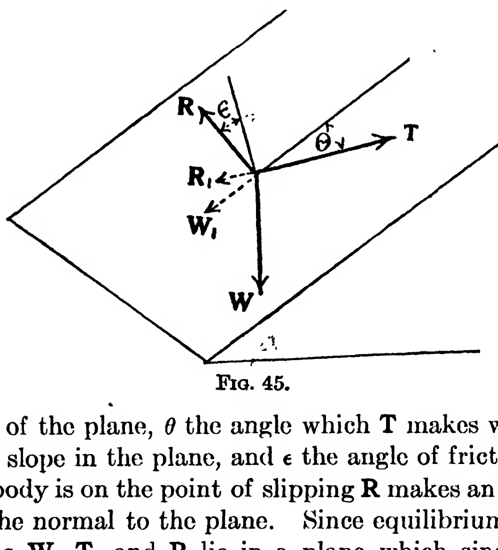


FIG. 45.

inclination of the plane, θ the angle which **T** makes with the line of greatest slope in the plane, and ϵ the angle of friction, so that when the body is on the point of slipping **R** makes an angle equal to ϵ with the normal to the plane. Since equilibrium exists, the three forces **W**, **T**, and **R** lie in a plane which since it passes through **W** is a vertical plane; but the normal to the inclined plane does not lie in it unless θ is zero or π.

From the resolution of the three forces along the normal, it is found that

$$R \cos \epsilon = W \cos \alpha.$$

Let $\mathbf{R}_1$ be the component of **R** in the inclined plane, so that

$$R \sin \epsilon = R_1 = W\mu \cos \alpha,$$

and let the angle which it makes with the line of greatest slope be denoted by ω. Let $\mathbf{W}_1$ be the component of **W** in the inclined

plane; then $\mathbf{W}_1$ lies in the line of greatest slope pointing down the plane, and

$$W_1 = W \sin \alpha.$$

The three forces $\mathbf{T}$, $\mathbf{W}_1$, $\mathbf{R}_1$ lie in the inclined plane and are in equilibrium. Also

$$\widehat{\mathbf{W}_1\mathbf{R}_1} = \pi - \omega, \qquad \widehat{\mathbf{R}_1\mathbf{T}} = \theta + \omega, \qquad \widehat{\mathbf{T}\mathbf{W}_1} = \pi - \theta.$$

Therefore,

$$\frac{R_1}{\sin \theta} = \frac{W_1}{\sin (\theta + \omega)} = \frac{T}{\sin \omega}$$

and

$$R_1 = \frac{W_1 \sin \theta}{\sin (\theta + \omega)} = W\mu \cos \alpha;$$

from which is derived

$$\sin (\theta + \omega) = \frac{1}{\mu} \tan \alpha \sin \theta,$$

an equation which determines ω if θ is given. If an infinitesimal increment is made to T, slippage will occur in the direction opposite to $\mathbf{R}_1$ which is the frictional component: that is, the direction of the motion is $\omega + \pi$.

118. The Friction of a Rope on a Curved Surface.—Imagine a rope or belt in contact with a rough cylindrical surface, not neces-

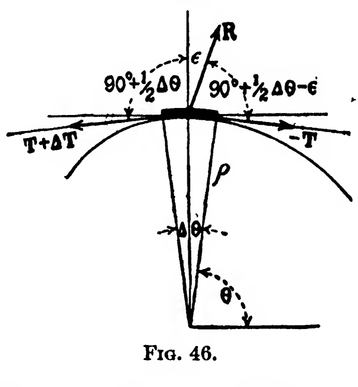


FIG. 46.

sarily circular, with the coefficient of friction μ. The tension on one end of the rope is greater than on the other and the rope is on the point of slipping. It is required to find the pressure of the rope upon the surface and the manner in which the change in the tension depends upon the coefficient of friction.

Let Δs be an element of the rope which subtends an angle $\Delta\theta$ at the center of curvature, with ρ as the radius of curvature. Let the tensions which act upon the two ends of Δs, due to the adja-

cent elements of the rope, be $-\mathbf{T}$ and $\mathbf{T} + \Delta\mathbf{T}$, and let $\mathbf{R}$ be the reaction of the surface. If the rope is about to slip, $\mathbf{R}$ makes an angle ϵ with the normal to the surface. Then

$$\frac{R}{\sin(\pi - \Delta\theta)} = \frac{-T}{\sin\left(90 + \epsilon + \frac{1}{2}\Delta\theta\right)} = \frac{T + \Delta T}{\sin\left(90 - \epsilon + \frac{1}{2}\Delta\theta\right)},$$

$$= \frac{\Delta T}{\sin\left(90 - \epsilon + \frac{1}{2}\Delta\theta\right) - \sin\left(90 + \epsilon + \frac{1}{2}\Delta\theta\right)},$$

or

$$\frac{R}{\sin(\pi - \Delta\theta)} = \frac{\Delta T}{2 \sin \epsilon \sin \frac{1}{2}\Delta\theta}.$$

On passing to the limit, these equations become

$$\frac{R}{d\theta} = \frac{T}{\cos \epsilon} = \frac{1}{\sin \epsilon} \cdot \frac{dT}{d\theta}.$$

Hence,

$$\frac{dT}{T} = \mu d\theta \quad \text{and} \quad T = T_0 e^{\mu\theta},$$

where T_0 is the value of the tension where the rope first touches the surface and θ is the angle through which the normal has turned from this point. It will be observed that this result is independent of the equation of the surface, although obviously the surface must be everywhere convex toward the rope. If μ is zero, the tension of the rope is constant.

If $\mathbf{P}$ is the normal pressure of the rope per unit length and $\mathbf{N}$ is the normal component of $\mathbf{R}$, then

$$N = R \cos \epsilon = P ds.$$

Hence,

$$T d\theta = P ds,$$

and therefore,

$$P = T\frac{d\theta}{ds} = \frac{T}{\rho};$$

that is, the pressure per unit length is directly proportional to the tension and inversely proportional to the radius of curvature.

119. Effect of a Snubbing Post.—If a horse can pull 1500 pounds on a rope and a man can hold 100 pounds, and if $\mu = 1/4$, how many times will it be necessary to wrap the rope around a snubbing post in order that the man may be able to hold the horse?

On setting $T_0 = 100$, $T = 1500$, and $\mu = 1/4$, in the formula for the tension, it is found that

$$\begin{aligned}\theta &= 4 \log_{\text{nat.}} 15. \\ &= 10.8322 = 1.72 \times 2\pi.\end{aligned}$$

The man will require at least one and three-fourths coils of the rope around the post.

Problems VIII

1. How high can a bug crawl up the inside of a hemispherical bowl if his coefficient of friction with the bowl is 0.25?

2. If a man can exert a horizontal pull of 200 lb., how much of a vertical pull must a crane apply to a stone that weighs 1000 lb. before the man can move the stone, the coefficient of friction being 1/3? *Ans.* 400 lb.

3. A body, resting on an inclined plane which makes an angle of 30° with the horizontal, is acted on by a horizontal force which is equal to the weight of the body. If the body is just on the point of slipping up the plane, what is the coefficient of friction? *Ans.* $\mu = 2 - \sqrt{3}$.

4. A weight W is sustained on a smooth inclined plane by three forces in the same vertical plane, each equal to $W/3$; one is directed toward the zenith, the second up the plane, and the third is horizontal. What is the inclination α of the plane? *Ans.* $\alpha = 2 \tan^{-1} 1/2$.

5. A smooth pulley is placed at the edge of a horizontal table. To a weight W resting on the table is attached a string which passes over the pulley and supports a weight w. Assuming that equilibrium exists and that the angle of friction is ϵ, through what angle α can the table be tipped before W begins to slip? *Ans.* $\alpha = \epsilon - \sin^{-1} (w \cos \epsilon/W)$.

6. Two weights $W_1 = 3$ lb. and $W_2 = 5$ lb. are tied to a small ring by means of inelastic strings and are then placed upon a rough horizontal table for which $\mu = 0.25$. When the strings are taut the angle between them is 60°. A third string, on which the tension is T, is tied to the ring and its direction makes an angle α with the bisector of the angle between the other two strings. What are the values of T and α if both weights are on the point of slipping? *Ans.* $T = 7/4 = 1.75$ lb.; $\sin \alpha = 1/7$; $\alpha = 8° 13'$.

7. Two weights $w_2 > w_1$ connected by a string rest on the surface of a smooth horizontal cylinder with the string on the surface of the cylinder in a plane perpendicular to its axis. What angle does the chord joining the two weights make with the vertical if the length of the string is one-fourth of the circumference of the cylinder? *Ans.* $\pi/4 + \tan^{-1} w_2/w_1$.

8. A body of weight W is sustained on a smooth inclined plane by two forces each equal to $W/2$, one acting horizontally and the other along the plane. What is the inclination of the plane? *Ans.* $\alpha = \sin^{-1} 4/5$.

9. A weight on a rough inclined plane is acted upon by a force directed up the plane, and is on the point of slipping down. If the angle of inclina-

tion 30° of the plane be doubled and the force also doubled, the weight is on the point of slipping upward. What is the coefficient of friction? *Ans.* $\mu = (5\sqrt{3} - 8)/11$.

10. The inclination of a smooth plane to the horizontal is $\alpha < 45°$. If a horizontal force of magnitude T sustains a body on the inclined plane, what other force of magnitude T also will sustain it? Compare the reactions of the plane in the two cases. *Ans.* $R_1/R_2 = \cos 2\alpha$.

11. Two weights are supported on a double inclined rough plane by means of a string which connects them, which passes over a smooth pulley in the ridge of the two planes and lies in a vertical plane perpendicular to the ridge. The two weights are on the point of slipping. Show that if the planes are tipped through an angle 2ϵ the weights again are on the point of slipping.

12. If the two planes in problem 11 make angles α and β with the horizontal, and if the two weights are w_1 and w_2, with w_2 on the point of slipping down, prove that the greatest weight that can be added to w_1 without disturbing the equilibrium is

$$\frac{\sin(\alpha + \beta)\sin 2\epsilon}{\sin(\alpha - \epsilon)\sin(\beta - \epsilon)}w_1.$$

13. A body is supported on a rough inclined plane by a force acting along it. If the least magnitude of the force, when the plane has an angle α with the horizontal, is equal to the greatest magnitude when the plane is inclined at an angle β, show that the angle of friction is $\epsilon = (\alpha - \beta)/2$.

14. If, in problem 7, the string passes along the chord instead of along the surface and if this chord subtends an angle 2α at the center of the cylinder, the inclination of the chord θ with the vertical is given by

$$\tan\theta = \frac{w_2 + w_1}{w_2 - w_1}\cot\alpha.$$

15. Two rings, each of weight w_1, are free to slide on a rough horizontal pole, the coefficient of friction being μ. The rings are connected by a string of length l which supports a smooth ring of weight $2w_2$. What is the greatest possible separation of the rings without sliding? *Ans.*

$$\frac{l(w_1 + w_2)\mu}{\sqrt{(w_1 + w_2)^2\mu^2 + w_2^2}}.$$

16. Two weights of different materials are laid on a rough plane, the inclination of which to the horizontal is α, and are connected by a taut string which makes an angle of 45° with the line of greatest slope of the plane. The lower weight is twice as heavy as the upper but the coefficient of friction of the upper weight, 2μ, is twice that of the lower. What is the value of μ if both weights are on the point of slipping? *Ans.* $\mu = \sqrt{5/8}\tan\alpha$.

17. Two rings of weights w_1 and w_3 are connected by a string and slide on two fixed smooth wires in the same plane, the former of which is vertical and the latter inclined at an angle α_3 with the horizontal. A weight w_2 is tied to the string and, when equilibrium is established, the two portions of the string make angles α_1 and α_2 with the vertical. Prove that

$$\frac{\cot\alpha_1}{w_1} = \frac{\cot\alpha_2}{w_1 + w_2} = \frac{\cot\alpha_3}{w_1 + w_2 + w_3}.$$

18. A heavy ring of weight w is free to slide on a smooth elliptic wire in a vertical plane. The eccentricity of the ellipse is e, and the major axis makes an angle α with the horizontal. A string fastened to the ring passes over a smooth peg at the center of the ellipse and supports a body also of weight w. Show that, when equilibrium is established, the angle φ which the tangent to the ellipse at the ring makes with the horizontal satisfies the equation

$$\tan(\varphi + \alpha) = (1 - e^2)\tan(2\varphi + \alpha).$$

19. A small ring slides freely upon a smooth elliptical hoop which is in a vertical plane with the major axis horizontal. It is acted on by two forces $\mathbf{F}_1$ and $\mathbf{F}_2$ directed toward the two foci and by $\mathbf{F}_0$ directed toward the center. Find the distance of the position of equilibrium from the center of the ellipse. *Ans.*

$$r = \frac{a\sqrt{1 - e^2}}{\sqrt{1 - \left(\frac{F_1 - F_2}{F_0}\right)^2}}.$$

20. A small ring slides freely upon a wire bent into a curve whose equation in bipolar coordinates is $r_1 r_2 = a$, where a is a constant. It is acted upon by two forces which are directed toward the two poles and such that

$$F_1 = \frac{k^2}{r_1} \quad \text{and} \quad F_2 = \frac{k^2}{r_2},$$

where k^2 is constant. Show that the ring is always in equilibrium.

21. Two weights W_1 and W_2 connected along the surface by a string of length l, rest upon the convex side of a smooth cycloidal cylinder, the radius of the generating circle of the cycloid being a. If θ is the angle between the vertical and the radius of the generating circle which corresponds to the position of W_2, show that

$$\sin\frac{1}{2}\theta = \frac{W_1}{W_1 + W_2}\cdot\frac{l}{4a}.$$

22. Two weights W_1 and W_2 rest on the concave side of a smooth parabola which has a horizontal axis, a vertical plane, and a latus rectum equal to $4p$. The two weights are connected by a string of length $l \cdot p$ which passes over a smooth peg at the focus. If θ is the angle which the string to W_1 (the higher of the two weights) makes with the axis, and $R = W_2/W_1$, show that

$$\tan\frac{1}{2}\theta = \sqrt{\frac{1 + R^2}{l - 2}}.$$

23. A weight is tied to two strings of length l_1 and l_2 and the other ends of the strings are attached to two points not necessarily at the same level. If the weight hangs in equilibrium, the horizontal components of the two tensions are equal, say equal to H. If the lengths of the strings are varied in such a way that H remains constant, prove that the weight describes a parabola which passes through the two points of suspension and has its axis vertical.

CHAPTER VIII

STATICS OF RIGID BODIES

I. THE DISPLACEMENTS OF RIGID BODIES

120. Definition of a Rigid Body.—A *rigid body* is a body in which the mutual distances of the particles remain invariable for every system of forces which can be applied to the body.

No physical body satisfies this definition. A rigid body in the mechanical sense is a mathematical fiction, which is convenient for the reason that many physical bodies act sensibly like rigid bodies if the applied forces are not too great. Such bodies are ordinarily called rigid because the deformations which they undergo escape detection. In the present chapter it will be supposed that the bodies dealt with are rigid in the mechanical sense of the word, unless something to the contrary is stated.

121. Definition of a Translation.—Suppose a fixed rectangular set of axes x, y, and z, which will be called a *trihedron*, is given to which the motion of the rigid body is referred. Imagine a second trihedron ξ, η, ζ which is rigidly attached to the body with the axes ξ, η, and ζ parallel to the corresponding x-, y-, and z-axes of the trihedron of reference.

If the body moves in such a way that the axes of the trihedron which is attached to the body are always parallel to the axes of reference, the motion is said to be a *pure translation*. It is characteristic of a pure translation that every particle of the body has the same velocity and the same acceleration as every other particle, the speeds and directions of motion being the same for all. The curves described by the particles form a system of parallel curves.

122. Definition of a Rotation.—If a body moves in such a way that two of its particles remain fixed with respect to the trihedron of reference, its motion is one of *pure rotation* about an axis which passes through the two fixed particles. Evidently each of the particles of the body describes a circle about the *axis of rotation*, the radius of the circle being equal to the perpendicular distance

of the particle from the axis. The speeds of the particles are proportional to the radii of the circles which they describe. Only those particles which lie on the same straight line parallel to the axis of rotation have the same velocity.

123. Displacements of a Rigid Body.—Any change of position of a rigid body is called a *displacement*. *Any displacement of a rigid body can be effected by a pure translation and a pure rotation in infinitely many ways.*

In order to prove this statement, let O be any particle of the body and S a sphere of any convenient radius with O as its center. Let A and B be any two particles of the body which lie on the sphere S. It is sufficiently evident, without argument, that by a pure translation of the body the particle O can be brought from its first position into coincidence with its second position. After this translation has been effected, the particles A and B will occupy certain positions upon the sphere S which will be denoted by A_1 and B_1. In the second position of the body, they also lie on the sphere S in certain other positions which will be denoted by A_2 and B_2. It remains to be shown that the particles A and B can be brought simultaneously from A_1 and B_1 to A_2 and B_2 by a pure rotation about an axis which passes through the point O; for, if three non-collinear points of the body be brought into coincidence, then the entire body comes into coincidence with its second position.

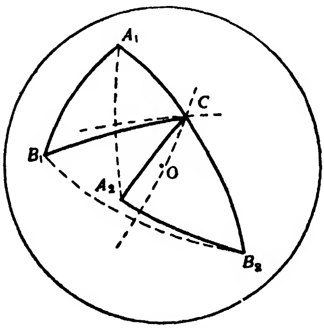


FIG. 47.

Pass a great circle through A_1 and B_1, and a second great circle through A_2 and B_2. Since the body is rigid,

$$\text{arc } A_1B_1 = \text{arc } A_2B_2.$$

Pass a third great circle through the points A_1 and A_2, and a fourth through B_1 and B_2. Bisect the arc A_1A_2 by a fifth great circle perpendicularly. Bisect the arc B_1B_2 by a sixth great circle perpendicularly. The fifth and sixth great circles intersect at a point C and in a second point C_1 diametrically opposite to C. The diameter CC_1 which, of course, passes through O is the axis of rotation which is sought. Since C lies on the perpendicular

bisector of the arc A_1A_2, it is equally distant from A_1 and A_2, that is,

$$\text{arc } A_1C = \text{arc } A_2C;$$

similarly,

$$\text{arc } B_1C = \text{arc } B_2C.$$

Hence, the spherical triangles A_1CB_1 and A_2CB_2 are equal, since the three sides of the one are equal, respectively, to the three sides of the other. Consequently, if the body is rotated about the axis COC_1 through the angle A_1CA_2 the point A_1 will be brought into coincidence with A_2 and, since the triangle A_1CB_1 equals the triangle A_2CB_2, the point B_1 comes into coincidence with B_2 at the same time. Since three non-collinear points have been brought into coincidence with the corresponding points of the second position, the entire body has been brought into coincidence. Thus the given displacement has been effected by a pure translation and a pure rotation. Since the point O and also the path of translation were arbitrary, the displacement could be effected in infinitely many ways.

124. Coplanar Displacements.—Let A, B, and C be three particles of a body lying in a plane π_1 but not in a straight line, and let D be a fourth particle in a plane π_2 parallel to π_1. If the displacement is of such a nature that A, B, and C continue to lie in the plane π_1 and D in the plane π_2, then the displacement is said to be coplanar; and if during a state of motion the points A, B, and C continue to lie in the plane π_1, the motion is said to be coplanar.

A coplanar displacement can always be effected by a pure rotation without translation and, furthermore, there is but one way in which it can be done. The axis of rotation will not, in general, lie within the body; it may even recede to infinity, in which case the rotation degenerates into a pure translation. This shows that a translation can be regarded as a rotation about an infinitely distant axis.

The proof is so similar to the proof for rotation in the preceding section that it will be left to the student.

125. The Transmissibility of Force.—In Sec. 108, it was shown that a string in equilibrium can be regarded as transmitting a force from one end to the other unaltered, since each particle is in equilibrium under the action of two forces only, and therefore the force acting on the last particle is the same as that acting on the first.

Other particles can be attached to the string without altering the equilibrium or the two forces acting on its extremities. In fact, as many as is desired can be added. Thus a body of finite width and thickness can be built up which continues in equilibrium and which transmits the force just as the string did, but it is not possible to analyse the forces acting on the individual particles. As the proof cannot be made precise at this point, the validity of the statement, *a rigid body acted on by two equal and opposite forces in the same straight line is in equilibrium,* must be assumed. This is commonly known as the principle of the *transmissibility of force.* It is evident that it makes no difference at what points in the line the forces are applied. It is essential, however, that the two forces shall lie in the same straight line, and that they shall be equal and opposite.

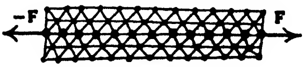


FIG. 48.

126. Theorem I.—*Two forces* $\mathbf{F}_1$ *and* $\mathbf{F}_2$ *in the same plane and such that*

$$\mathbf{F}_1 + \mathbf{F}_2 \neq 0,$$

acting at the points A and B of a rigid body, are equivalent to a single force

$$\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2,$$

which lies in the same plane and which has a certain definite line of action.

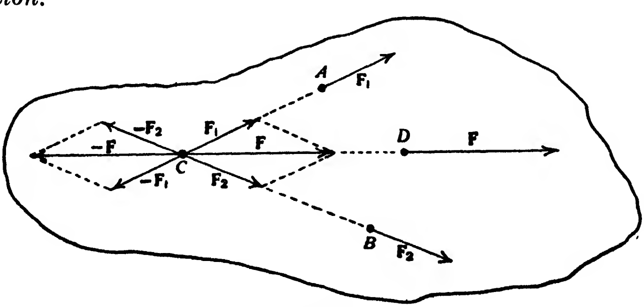


FIG. 49.

Let $\mathbf{F}_1$ and $\mathbf{F}_2$ act at the points A and B, respectively. Let their line of action be continued until they intersect in the point C. Let two pairs of equal but opposite forces $\mathbf{F}_1$ and $-\mathbf{F}_1$ and $\mathbf{F}_2$ and $-\mathbf{F}_2$ be introduced at C. These forces will have no effect upon the particle at C and, therefore, no effect upon the rigid

body. By the principle of the transmissibility of force, the body is in equilibrium under the actions of the two forces $\mathbf{F}_1$ acting at A and $-\mathbf{F}_1$ acting at C; and also in equilibrium under the actions of $\mathbf{F}_2$ acting at B and $-\mathbf{F}_2$ acting at C. There remain then the forces $\mathbf{F}_1$ and $\mathbf{F}_2$ acting on the particle at C. But $\mathbf{F}_1$ and $\mathbf{F}_2$ acting on a particle are equivalent to their resultant

$$\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2$$

acting on the same particle.

Now let the forces $-\mathbf{F}$ and $+\mathbf{F}$ be introduced at the points C and D, respectively, where D is any point in the line of $\mathbf{F}$ through C. The introduction of these forces will have no effect upon the body. The forces $\mathbf{F}$ and $-\mathbf{F}$ acting at C are in equilibrium, and there remains only the force $\mathbf{F}$ acting at the point D.

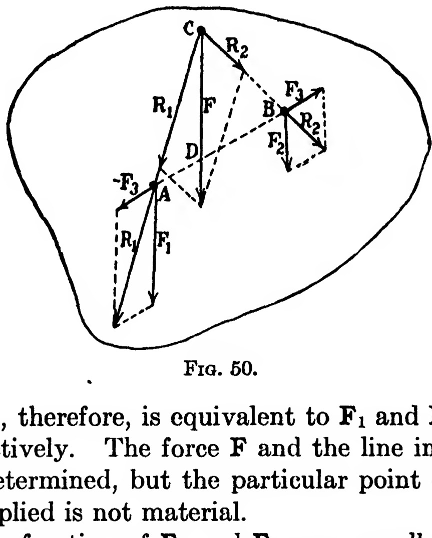


FIG. 50.

$\mathbf{F}$ acting at D, therefore, is equivalent to $\mathbf{F}_1$ and $\mathbf{F}_2$ acting at A and B, respectively. The force $\mathbf{F}$ and the line in which it acts is uniquely determined, but the particular point of that line at which it is applied is not material.

If the lines of action of $\mathbf{F}_1$ and $\mathbf{F}_2$ were parallel, the point C would not exist. This difficulty is avoided by introducing two forces $\mathbf{F}_3$ and $-\mathbf{F}_3$ acting at B and A, respectively, in the line AB (Fig. 50). Now $\mathbf{F}_1 - \mathbf{F}_3 = \mathbf{R}_1$ acting at A, and $\mathbf{F}_2 + \mathbf{F}_3 = \mathbf{R}_2$ acting at B, so that, the system of forces $\mathbf{F}_1$ and $\mathbf{F}_2$ acting at A and B, respectively, is equivalent to the system $\mathbf{R}_1$ and $\mathbf{R}_2$ acting at the same points. But, if

$$\mathbf{F}_1 + \mathbf{F}_2 \neq 0, \qquad \text{then} \qquad \mathbf{R}_1 + \mathbf{R}_2 \neq 0.$$

The lines of action of $\mathbf{R}_1$ and $\mathbf{R}_2$ intersect at some point C, and the forces $\mathbf{R}_1$ and $\mathbf{R}_2$ can be slid along their lines of action to the point

C. Let $\mathbf{F}$ be their resultant acting at C. Then $\mathbf{F}$ is the equivalent of the original pair of forces $\mathbf{F}_1$ and $\mathbf{F}_2$ acting at A and B, respectively. Furthermore,

$$\mathbf{F} = \mathbf{R}_1 + \mathbf{R}_2 = (\mathbf{F}_1 - \mathbf{F}_3) + (\mathbf{F}_2 + \mathbf{F}_3) = \mathbf{F}_1 + \mathbf{F}_2$$

and, since $\mathbf{F}_1$ and $\mathbf{F}_2$ are parallel to each other, $\mathbf{F}$ likewise is parallel to each of them. Therefore,

$$F = F_1 + F_2, \qquad \text{or} \qquad F = |F_1 - F_2|,$$

according as $\mathbf{F}_1$ and $\mathbf{F}_2$ have the same or opposite directions. where $|F_1 - F_2|$ means the numerical value of the difference.

If D is the point where the line of $\mathbf{F}$ intersects the line AB, and if $\mathbf{F}_1$ and $\mathbf{F}_2$ have the same direction, the point D will lie between A and B, and

$$\frac{\overline{AD}}{\overline{DB}} = \frac{F_2}{F_1}.$$

But if $\mathbf{F}_1$ and $\mathbf{F}_2$ have opposite directions the point D lies outside of the interval $\overline{AB}$.

127. Theorem II.—*The n forces* $\mathbf{F}_1, \mathbf{F}_2, \cdots, \mathbf{F}_n$ *in the same plane and such that*

$$\mathbf{F}_1 + \mathbf{F}_2 + \cdots + \mathbf{F}_n \neq 0,$$

acting at the points $A_1, A_2, \cdots, A_n$, *respectively, of a rigid body, are equivalent to a single force*

$$\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 + \cdots + \mathbf{F}_n$$

which lies in the same plane and has a certain definite line of action.

This theorem is a generalization of theorem I which is immediately apparent. By combining any two of the forces into a single one, the number of forces is reduced from n to $n - 1$ and a repetition of the process leads eventually to a single force with a definite line of action.

If the forces are all parallel, the resulting single force will be parallel to the original system of forces and its magnitude will be the sum, in the algebraic sense, of the magnitudes of the original forces.

128. The Equivalent of a System of Coplanar Forces—Graphical Construction.—Let the plane system of forces $\mathbf{F}_1, \mathbf{F}_2, \cdots, \mathbf{F}_5$ be acting upon a rigid body along lines as indicated in Fig. 51. It is desired to find the single force $\mathbf{F}$ and its line of action which is the equivalent of the given system. The resultant force $\mathbf{F}$ is

obtained by constructing the polygon of forces (magnified in the ratio 1:2 in the diagram), that is, by placing the vectors end to end and then joining the origin of the first vector to the terminus of the last.

In order to obtain the line of action of **F**, take any point O in the plane of the polygon of forces and join it to each of the vertices of the polygon. Let these lines be regarded as vectors with origins at the point O and termini at the vertices of the polygon.

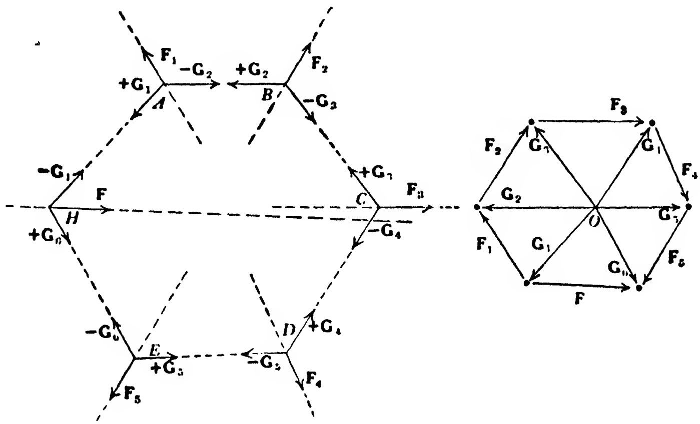


FIG. 51.

Let the vector to the origin of $\mathbf{F}_1$ be $\mathbf{G}_1$, to the origin of $\mathbf{F}_2$ be $\mathbf{G}_2$, and so on, the last one $\mathbf{G}_6$ being drawn to the terminus of $\mathbf{F}_5$. Then the following relations are evident from the closed triangles:

$$\left.\begin{array}{ll}\mathbf{G}_1 + \mathbf{F}_1 - \mathbf{G}_2 = 0, & \mathbf{G}_4 + \mathbf{F}_4 - \mathbf{G}_5 = 0,\\ \mathbf{G}_2 + \mathbf{F}_2 - \mathbf{G}_3 = 0, & \mathbf{G}_5 + \mathbf{F}_5 - \mathbf{G}_6 = 0,\\ \mathbf{G}_3 + \mathbf{F}_3 - \mathbf{G}_4 = 0, & \mathbf{G}_6 - \mathbf{F} - \mathbf{G}_1 = 0.\end{array}\right\} \quad (1)$$

Through any point A of the rigid body on the line of action of $\mathbf{F}_1$, draw a line parallel to $\mathbf{G}_1$ and also draw a line parallel to $\mathbf{G}_2$. Let the intersection of this second line with the line of action of $\mathbf{F}_2$ be B. From B draw a line parallel to $\mathbf{G}_3$ intersecting the line of action of $\mathbf{F}_3$ at C. Let this process be continued until finally the line parallel to $\mathbf{G}_6$ intersects the line parallel to $\mathbf{G}_1$ at a point H. Then the line of action of the equivalent force **F** passes through H.

To prove this, let the forces $-\mathbf{G}_1$ and $+\mathbf{G}_1$ be introduced at H and A, respectively; $-\mathbf{G}_2$ and $+\mathbf{G}_2$ at A and B; $-\mathbf{G}_3$ and $+\mathbf{G}_3$ at B and C; and so on, until finally $-\mathbf{G}_6$ and $+\mathbf{G}_6$ are introduced at E and H, respectively. The forces thus introduced have no effect on the rigid body, since they occur in pairs which are equal and opposite and lie in the same straight line. But since, by Eq. (1),

$$\mathbf{G}_1 + \mathbf{F}_1 - \mathbf{G}_2 = 0,$$

the forces acting at A are in equilibrium; and likewise those at B, C, D, and E are in equilibrium. Only at the point H are the forces not in equilibrium; and since, by the last equation of Eq. (1),

$$\mathbf{G}_6 - \mathbf{G}_1 = \mathbf{F},$$

the given system of forces $\mathbf{F}_1, \mathbf{F}_2, \cdots, \mathbf{F}_5$ are equivalent to the single force $\mathbf{F}$ acting at H.

129. The Center of Gravity.—The attraction of the earth acts upon each particle of a body with a force

$$\mathbf{W} = m\mathbf{g},$$

where m is the mass of the particle and $\mathbf{g}$ is the acceleration of gravity which is the same for all particles. The line of action of the equivalent single force is sought.

Let m_1 and m_2 be the masses of two particles at A and B, respectively, and let $\mathbf{W}_1$ and $\mathbf{W}_2$ be their weights. Let P be the point on the line AB through which passes the line of action of the equivalent single force $\mathbf{W}_1 + \mathbf{W}_2$. Then, by Sec. 126,

$$\frac{AP}{PB} = \frac{W_2}{W_1} = \frac{m_2}{m_1}.$$

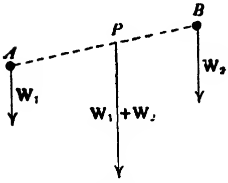


FIG. 52.

It will be observed that the position of the point P is independent of the direction of $\mathbf{g}$. It depends only upon the ratio of the masses.

Let x_1, y_1, and z_1 be the coordinates of the point A; x_2, y_2, and z_2 the coordinates of B; and $\bar{x}$, $\bar{y}$, and $\bar{z}$ the coordinates of P. Let λ, μ, and ν be the angles which the line AB makes with the x-, y-, and z-axes, respectively. Then

$$\frac{\overline{AP}}{\overline{BP}} = \frac{\overline{AP} \cdot \cos\lambda}{\overline{BP} \cdot \cos\lambda} = \frac{\bar{x} - x_1}{x_2 - \bar{x}} = \frac{m_2}{m_1},$$

and, on solving for $\bar{x}$,

$$\left.\begin{aligned} \bar{x} &= \frac{m_1x_1 + m_2x_2}{m_1 + m_2}; \\ \text{similarly,} \qquad \bar{y} &= \frac{m_1y_1 + m_2y_2}{m_1 + m_2}, \\ \bar{z} &= \frac{m_1z_1 + m_2z_2}{m_1 + m_2}. \end{aligned}\right\} \tag{1}$$

Let the single force $\mathbf{W}_1 + \mathbf{W}_2$ acting at the point P be combined with the weight $\mathbf{W}_3$ of a third particle of mass m_3 and coordinates x_3, y_3, and z_3. It is not necessary to repeat the argument since Eq. (1) is now available. Thus, it is found for three particles that

$$\begin{aligned} \bar{x} &= \frac{(m_1 + m_2)\dfrac{m_1x_1 + m_2x_2}{m_1 + m_2} + m_3x_3}{(m_1 + m_2) + m_3}, \\ &= \frac{m_1x_1 + m_2x_2 + m_3x_3}{m_1 + m_2 + m_3}; \end{aligned}$$

and similar expressions for $\bar{y}$ and $\bar{z}$.

By mathematical induction, it is found in general that

$$\left.\begin{aligned} \bar{x} &= \frac{m_1x_1 + m_2x_2 + \cdots + m_nx_n}{m_1 + m_2 + \cdots + m_n} = \frac{\Sigma m_ix_i}{\Sigma m_i}, \\ \bar{y} &= \frac{m_1y_1 + m_2y_2 + \cdots + m_ny_n}{m_1 + m_2 + \cdots + m_n} = \frac{\Sigma m_iy_i}{\Sigma m_i}, \\ \bar{z} &= \frac{m_1z_1 + m_2z_2 + \cdots + m_nz_n}{m_1 + m_2 + \cdots + m_n} = \frac{\Sigma m_iz_i}{\Sigma m_i}. \end{aligned}\right\} \tag{2}$$

The point whose coordinates are $\bar{x}$, $\bar{y}$, and $\bar{z}$ is called the *center of gravity* of the body. It will be observed that it is independent of the direction of gravity. It is also the centroid of the points x_i, y_i, and z_i for the weighting factors m_i, (Sec. 21) and is therefore a fixed point of the body which is independent of the coordinate system. It is also called the *center of mass*.

However the body may be oriented the line of action of the equivalent single force passes through the center of gravity, and the magnitude of the equivalent force is the weight of the body. It is for this reason that it is said that *the weight of a rigid body acts at its center of gravity.*

130. Theorem III.—*The work done in raising a rigid body against gravity is equal to the weight of the body multiplied by the height through which its center of gravity is raised.*

Let m_i be the mass of the ith particle, h_i its height before the body is raised, and h_i^* its height after the body has been raised. Since no work is done upon the particle against gravity by a horizontal displacement, the total work done upon the particle is $m_ig(h_i^* - h_i)$. The work done on the whole body is the sum of the work done on the separate particles. Hence,

$$\begin{aligned} \text{total work } &= \sum m_ig(h_i^* - h_i), \\ &= g(\sum m_ih_i^* - \sum m_ih_i), \\ &= g(MH^* - MH), \\ &= W(H^* - H), \end{aligned}$$

where

$$\begin{aligned} M &= \sum m_i = \text{total mass}, \\ W &= Mg = \text{total weight}, \\ H^* &= \text{height of center of gravity after raising}, \end{aligned}$$

and $$H = \text{height of center of gravity before raising.}$$

It will be observed that the property of rigidity was not used in the demonstration. The theorem holds, therefore for bodies or systems of bodies that are not rigid.

Since the work done in raising a body from one position to another is the potential energy of the body in the second position with respect to the first (Sec. 68), it follows that the potential energy of a body, whose center of gravity is at a height h above the ground and whose weight is w, is wh with respect to the ground.

Problems IX

1. A light rod, which passes through the centers of spheres of weights of 1, 2, 3, 4, and 5 lb., the distances of the centers being 1, 2, 3, 4, and 5 ft. from the end of the rod, is supported in a horizontal position by a single string. Where is the string attached to the rod, and what is its tension? *Ans.* 3 2/3 ft. from end of rod; 15 lb.

2. A uniform bar 5 ft. long weighing 120 lb. rests on two supports which are 10 and 20 in. from the ends of the bar. How much weight does each support carry? *Ans.* 40 and 80 lb.

3. A 30-lb. weight is placed on the end of the bar which has the closest support in problem 2. What are the weights now carried by each support? *Ans.* 80 and 70 lb.

4. A uniform bar 4 ft. long weighs 10 lb., and weights of 30 and 40 lb. are attached to its two extremities. At what point must the bar be supported in order that it may balance? *Ans.* 3 in. from the center.

5. A bar, each foot of which weighs 7 lb. rests upon a fulcrum 3 ft. from one extremity. What must be its length in order that a weight of 71 1/2 lb. suspended from that extremity may be just balanced by 20 lb. suspended from the other? *Ans.* 9 ft.

6. Equal parallel forces act at five of the corners of a regular hexagon whose diagonal is a. Show that the equivalent single force cuts the diagonal through the sixth corner at a distance of $3a/5$ from the corner.

7. A 10 ft. bar weighing 30 lb. rests on two supports which are 1 and 3 ft. from the ends of the bar. A string attached to the bar is pulled vertically upward until the supports each carry 12 lb. Where is the string tied and what is its tension? *Ans.* 1 ft. from end of bar; 6 lb.

8. A square plate of side 1 ft. cannot be placed upon three pegs at the three corners of an equilateral triangle of side 1 ft. so as to distribute its weight equally upon the three pegs. Show that the best that can be done is to distribute the weight in the ratios

$$w_1 : w_2 : w_3 : : \frac{1}{2\sqrt{3}} : \frac{1}{2\sqrt{3}} : 1 - \frac{1}{\sqrt{3}}.$$

9. A uniform chain passes over a pulley in the ridge formed by two smooth planes which make angles α and β with the horizontal. Prove that the chain will remain at rest if its two ends are on the same level.

10. The length of the beam of a false balance is 3 ft. 9 in. A body placed in one pan balances a 9-lb. weight in the other; but, when placed in the other, it balances a 4-lb. weight. Find the true weight w of the body and the lengths a and b of the arms. *Ans.* $w = 6$ lb.; $a = 1\ 1/2$ ft.; $b = 2\ 1/4$ ft.

11. If a balance is false, having its arms in the ratio of 15:16, find how much a customer really pays for an article which is sold to him from the longer arm at 75 cts. per pound. *Ans.* 80 cts.

12. A heavy horizontal plate is supported by five strings attached to the plate at points whose coordinates are $-1, +1$; $+1, +1$; $+1, -1$; $-1, -1$; and $0, 0$. The tensions in the strings are, respectively, 6, 7, 8, 9, and 10 lb. Where is the center of gravity of the plate? *Ans.* $\bar{x} = 0$; $\bar{y} = -1/10$.

II. MOMENTS OF VECTORS

131. The Concept of Moments.—If a force acts upon a particle, the only motion that can occur is a translation. The measure of the force is the product of mass of the particle and its acceleration. If, however, a force acts upon a rigid body, it may result in a translation, or in a translation and a rotation. If merely a translation occurs, the force is measured by the product of the mass of the rigid body and its acceleration, just as for a particle. If a rotation also occurs, the matter is somewhat more complicated and the concept of *moments* is required for its study.

132. The Moment of a Vector with Respect to a Point.—The moment of a vector with respect to a point is the product of the magnitude of the vector and the perpendicular distance from the point to the line of the vector.

Let the plane which passes through the vector and the point be taken as the xy-plane, with the point O as the origin of a rectangular set of coordinates.

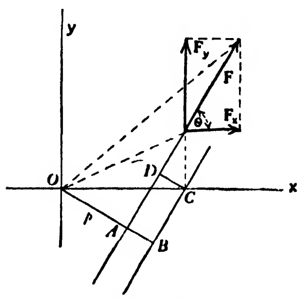


Fig. 53.

In Fig. 53, let **F** be the vector, O the given point, and p the perpendicular from O to the line of **F**. Then the moment of **F** with respect to O is $p \cdot F$, taken positively or negatively according as **F** indicates positive or negative motion about the point O. Since p is the altitude and F is the base of the triangle defined by the vector **F** and the point O, the product $p \cdot F$ is twice the area of this triangle. Let O-**F** denote the area of the triangle. Then

$$\text{moment} = 2 \cdot O\text{-}\mathbf{F}.$$

The moment vanishes if the vector vanishes, or if the line of the vector passes through O, that is, if p vanishes.

If the origin of **F** is at the point x, y and θ is the angle between **F** and the positive direction of the x-axis, then

$$p = x \sin \theta - y \cos \theta;$$

for

$$OB = x \sin \theta, \qquad AB = CD = y \cos \theta, \qquad p = OB - AB.$$

Hence,

$$p \cdot F = xF \sin \theta - yF \cos \theta,$$

or

$$p \cdot F = xF_y - yF_x,$$

from which it is seen that the moment of a vector is the algebraic sum of the moments of its components; for in the diagram it is seen that the moments of **F** and $\mathbf{F}_y$ are positive with respect to O, while the moment of $\mathbf{F}_x$ is negative.

133. The Moment of a Vector is Itself a Vector.—On the line through O perpendicular to the plane which contains the vector **F** and the point O, mark off a distance equal to $p \cdot F$ in the positive direction from O, if **F** is directed counterclockwise with respect to O, and in the negative direction, if **F** is directed clockwise

with respect to O. The directed line thus indicated (**M**, Fig. 54) represents in magnitude and direction the moment of **F** with respect to O.

In vector analysis, the vector product of two vectors **A** and **B** (Sec. 55) is defined as a vector **C** perpendicular to the plane which contains **A** and **B** and in magnitude

$$C = A \cdot B \cdot \sin \theta,$$

where θ is the angle between **A** and **B**. The vector product is written

$$\mathbf{C} = \mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A}. \tag{1}$$

The vector **C** is in the positive direction if the angle $\theta (< 180°)$ measured from **A** to **B** is counterclockwise, otherwise it is in the negative direction. This accounts for the negative sign in

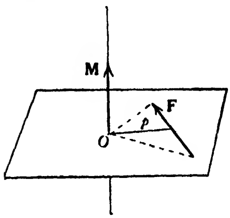


FIG. 54.

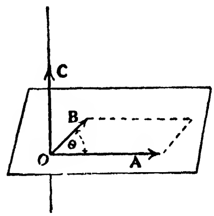


FIG. 55.

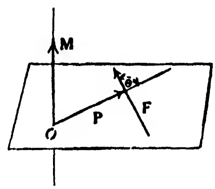


FIG. 56.

Eq. (1), since the angle from **B** to **A** is opposite in direction from the same angle measured from **A** to **B**.

Given a vector **F** and a point O (Fig. 56). From O as an origin draw **P**, any vector terminating in the line of **F**. The moment of **F** with respect to O is

$$\mathbf{M} = \mathbf{P} \times \mathbf{F},$$

since **M** is perpendicular to the plane of **P** and **F** and the magnitude of $\mathbf{P} \times \mathbf{F}$ is $P \cdot F \cdot \sin \theta = p \cdot F \qquad (P \sin \theta = p)$.

134. The Moment of a Vector with Respect to an Axis.—The moment of a vector with respect to an axis on which has been chosen a positive direction is the projection on that axis of the moment of the given vector with respect to any point of the axis. It is necessary, however, to show that the moment so defined is independent of the point chosen.

Let L be the given axis and **F** the given vector. Let O be any point on L and **G** the moment of **F** with respect to O. Let **M** be the projection of **G** on L. Then **M** is the moment of **F** with respect to L.

In order to prove that **M** is independent of the position of the point O, let P_1 be the plane through O which contains the vector **F** and let P_2 be the plane through O perpendicular to L. Let the angle of inclination between P_1 and P_2 be θ; then θ is also the angle between **G** and **M**, since they are perpendicular to P_1 and P_2, respectively.

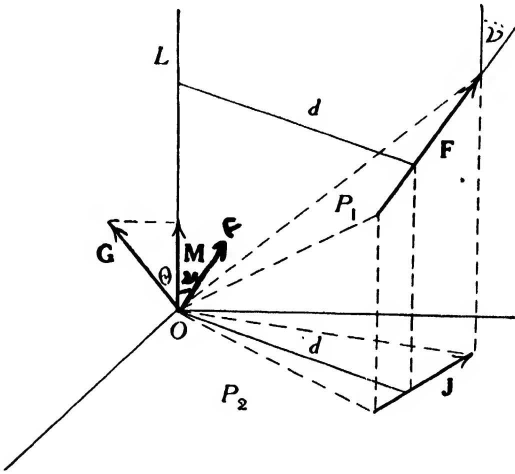


FIG. 57.

Let **J** be the projection of **F** on P_2. The area of the triangle O-**J** is independent of the position of O on L. But, by the projection of areas,

$$\begin{aligned} 2O\text{-}\mathbf{J} &= 2O\text{-}\mathbf{F} \cdot \cos\theta \\ &= G \cdot \cos\theta \\ &= M \end{aligned}$$

and, therefore, M also is independent of the position of the point O.

Let ν be the angle between the axis L and the line of **F**, and let d be the shortest line between L and the line of **F**. Then

$$J = F \cdot \sin\nu.$$

The projection of d upon the plane P_2 is d itself, since d and P_2 are both perpendicular to L. Hence,

$$2O\text{-}\mathbf{J} = M = d \cdot J = F \cdot d \cdot \sin\nu,$$

and it is seen that the moment of a vector with respect to an axis vanishes if, and only if, the vector itself vanishes, intersects the axis, or is parallel to the axis.

If the vector **F** is resolved into its three rectangular components, it is readily proved that the moment of **F** with respect to an axis is the vector sum of the moments of its components with respect to the same axis.

135. The Moment of a Force.—It is shown in dynamics, as a consequence of the laws of motion, that *the rate of change of angular momentum is equal to the sum of the moments of the forces acting;* or, in simpler language, the moment of a force with respect to an axis measures the effectiveness of the force in producing rotation. It cannot be proved here, and it must therefore be assumed.

It can be illustrated, however, very simply. Imagine a capstan with an arm of length a. At the end of the arm a force **F** is acting in some direction. Let **F** be resolved into three rectangular components; one, $\mathbf{F}_x$ along the arm; the second, $\mathbf{F}_y$ perpendicular to the plane which contains the arm and the axis of rotation; the third, $\mathbf{F}_z$ parallel to the axis of rotation. The moment of **F** is equal to the vector sum of the moments of its components. The moment of $\mathbf{F}_x$ is zero, since its line of action passes through the axis of rotation. It is also obvious that $\mathbf{F}_x$ is a mere pull (or push) on the arm and has no effect in turning the capstan.

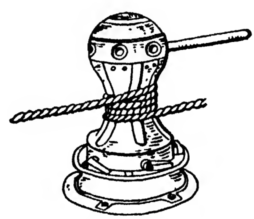


FIG. 58.

The moment of $\mathbf{F}_z$ also is zero, since it is parallel to the axis. The force $\mathbf{F}_z$ itself is a pull, up or down, on the end of the arm and evidently has no tendency to produce rotation. The only component which has any effect in producing rotation is $\mathbf{F}_y$ which is perpendicular to the plane through the axis of rotation and the point of application of the force. For a given length of arm, the turning effect is proportional to F_y; and a given F_y is more effective in producing rotation the longer the arm on which it works (the principle of the lever). That is to say, the product $a \cdot F_y$ measures the tendency of **F** to produce rotation, and $a \cdot F_y$ is precisely the moment of **F** with respect to the axis of rotation.

136. The Wheel and Axle.—A large wheel of radius R_1 is attached to an axle of radius R_2 with the axis of the axle passing through the center of the wheel. At the end of a rope which is wound about the axle is suspended a weight W_2. Likewise, a

weight W_1 is suspended from a rope which passes over the circumference of the wheel and is attached to it. The apparatus is supported by two horizontal pivots on the axis of the axle about which it is free to turn. The pivots are supposed to be frictionless and the ropes of negligible weight. Under what conditions will it be in equilibrium?

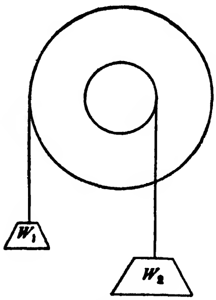


FIG. 59.

As represented in Fig. 59, the moment of the weight W_1 with respect to the axis of rotation is R_1W_1 and, if not opposed, would produce a counterclockwise rotation of the wheel and axle. The moment of the other weight is R_2W_2 and this would produce a clockwise rotation. The only other forces acting are the weight of the wheel and axle and the support of the pivots but, as their lines of action pass through the axis, they have no effect on the rotation. The condition for equilibrium is, therefore,

$$R_1W_1 = R_2W_2,$$

that is, the two opposing moments must be equal. If this condition is not satisfied, the wheel and axle will rotate in the direction of the larger moment. The smaller weight may actually make the larger weight rise.

137. A Horizontal Bar.—A uniform bar AB of weight W moves freely about the end B as on a hinge. It is supported in a horizonal position by a string AC which makes an angle of 45° with it. Find the tension in the string.

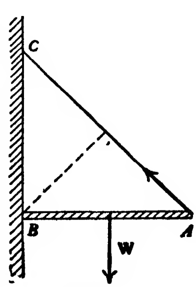


FIG. 60.

The bar is prevented from turning about B by two forces, the tension of the string $\mathbf{T}$ and the weight of the bar $\mathbf{W}$. Let the length of the bar be $2l$. The moment of $\mathbf{W}$ with respect to B is $l \cdot W$, since the weight of the bar acts at its center. The moment of $\mathbf{T}$ with respect to B is $p \cdot T$ and $p = \sqrt{2}l$. The reaction of the wall on the bar passes through B so that its moment is zero. There are no other forces acting on the bar and, therefore, since the bar is in equilibrium,

$$\sqrt{2}l \cdot T = l \cdot W, \quad \text{or} \quad T = \frac{W}{\sqrt{2}}.$$

138. A Three-legged Table.—The legs of a table are at the corners of an equilateral triangle. If a weight W is placed upon the table in a given position, what fraction of its weight is carried by each leg?

Let ABC be the equilateral triangle, of which each side is S, and let the position of W on the table be given by the lengths of the perpendiculars to the three sides of the triangle p_a, p_b, and p_c.

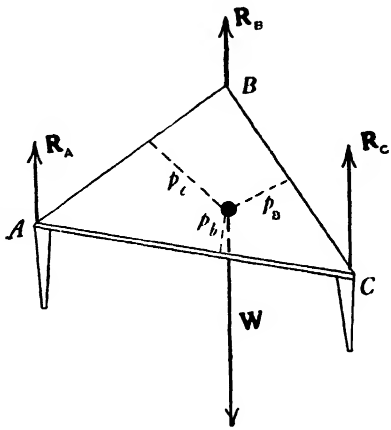


FIG. 61.

The table is in equilibrium without the weight W and it is also in equilibrium with the weight resting on it. The top of the table, therefore, is in equilibrium under the weight of W, $\mathbf{W}$, and the reaction of the three legs $\mathbf{R}_a$, $\mathbf{R}_b$, and $\mathbf{R}_c$, due to W, which are equal and opposite to the fractional part of the weight of W which each supports. On taking moments about the line BC, $\mathbf{R}_b$ and $\mathbf{R}_c$ are eliminated, since their moments are each zero. The remaining forces give

$$p_a \cdot W = R_A \cdot \frac{S\sqrt{3}}{2}.$$

Also,
$$p_b \cdot W = R_B \cdot \frac{S\sqrt{3}}{2},$$

and
$$p_c \cdot W = R_C \cdot \frac{S\sqrt{3}}{2},$$

by taking moments about AC and AB, respectively. Hence,

$$R_A : R_B : R_C = p_a : p_b : p_c,$$

together with
$$R_A + R_B + R_C = W$$

and
$$p_a + p_b + p_c = S\sqrt{3}/2.$$

If a fourth leg were fitted into the table somewhere, it is evident that this fourth leg might be made to carry no weight at all, or that one of the other three legs might be made to carry no weight at all, or that any other distribution of weight between these extremes might be made. The problem of a *rigid* table with

four legs is therefore indeterminate. The problem becomes determinate only when the table is deformable and this requires that certain properties of the particular table used shall be known, a very difficult condition, in general.

Problems X

1. A wheel, which is free to turn about a horizontal axis, has a weight of 2 lb. attached to the end of a spoke which makes an angle of 60° with the horizontal. What weight must be attached to the end of a horizontal spoke to keep the wheel in equilibrium?

2. Two men walking 10 ft. apart carry a 200-lb. weight suspended from a pole which they carry on their shoulders. If the weight is 4 ft. from one of the men, how much weight does each man carry?

3. A table of negligible weight has any number of legs, which are not necessarily vertical. How can a weight be placed on the table so that the table will tip?

4. The two ropes of a wheel and axle are tied together below the axle and a weight is tied to them. Show that, in equilibrium, the weight is directly below the axle.

5. A plank of negligible weight rests on a rough horizontal log of radius a which is fixed. Two persons of weight w_1 and w_2, $w_1 < w_2$, 15 ft. apart, are sitting on the plank which is just on the point of slipping. What is the inclination of the plank, and how far is each person from the point of contact of the plank with the log? How far must w_1 move back in order that the plank may be horizontal? *Ans.* $a\epsilon w_2/w_1$ ft.

6. AB is a lever and C is its fulcrum. Two forces $\mathbf{F}_A$ and $\mathbf{F}_B$ in the same plane act on the ends of the lever, making angles α and β ($> 90°$) with the lever. The reaction of the fulcrum on the lever makes an angle θ with the lever, show that, in equilibrium

$$\tan\theta = \frac{F_A\sin\alpha + F_B\sin\beta}{F_A\cos\alpha - F_B\cos\beta}.$$

7. Given two axles of the same diameter and their wheels which are provided with cogs. The first wheel has 60 cogs, the second 40 and they are in gear. If the first axle supports 160 lb., what must the second support? What effect will it have if a small pinion of 10 cogs is placed between the two wheels? *Ans.* 106 2/3 lbs.

8. A uniform bar 10 ft. long weighing 100 lb. is supported by two pegs 1 ft. apart, the end of the bar being just under one peg A and the bar passing over the other B. What is the reaction of each peg? *Ans.* $F_A = 400$ lb.; $F_B = 500$ lb.

III. THEOREMS RELATING TO COUPLES

139. Couples.—A system of two forces $\mathbf{F}_1$ and $\mathbf{F}_2$ not lying in the same straight line but such that $\mathbf{F}_1 + \mathbf{F}_2 = 0$ is called a *couple.* The two forces are parallel and equal in magnitude but oppositely

directed. The perpendicular distance between the two parallel lines of the forces is called the *arm of the couple*. A couple cannot be reduced to a single force as are other systems; for if, in the discussion of parallel forces under theorem I (Sec. 126), $\mathbf{F}_2$ is set equal to $-\mathbf{F}_1$, it results that $\mathbf{R}_2 = -\mathbf{R}_1$; their lines of action are still parallel and, therefore, do not intersect. The artifice of introducing two forces $\mathbf{F}_3$ and $-\mathbf{F}_3$ in the same straight line does not succeed. The difficulty is a real one and cannot be avoided by artifices.

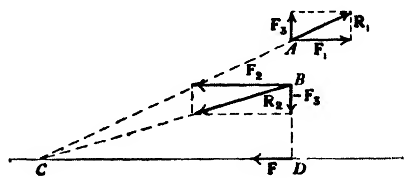


Fig. 62.

To show this, let $\mathbf{F}_1$ and $\mathbf{F}_2$ be parallel and oppositely directed, and let

$$F_2 = F_1 + F,$$

where F is small. Then, by theorem I, the resulting single force is $\mathbf{F}$ and its line of action is parallel to those of $\mathbf{F}_1$ and $\mathbf{F}_2$, intersecting the line AB at the point D (Fig. 62). If $\mathbf{F}_1$ remains fixed and $\mathbf{F}_2$ diminishes in magnitude so that F tends toward zero, its line of action remaining unaltered, the point C recedes to infinity along the line AC and the point D also recedes to infinity along the line AD. In the limit, for $F = 0$, the resultant force $\mathbf{F}$ also is zero but the arm on which it acts is infinitely long, so that its moment with respect to any finite point is not determined.

140. Theorem IV.—*The moment of a couple with respect to any axis perpendicular to its plane is independent of the position of the axis.*

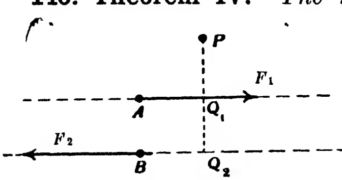


Fig. 63.

Let $\mathbf{F}_1$ and $\mathbf{F}_2$ be the two forces of the couple, and let a be its arm. The moment of the couple with respect to any point P is

$$\begin{aligned} \overline{PQ}_2 \cdot F_2 - \overline{PQ}_1 \cdot F_1 &= F_1(\overline{PQ}_2 - \overline{PQ}_1) \\ &= a \cdot F_1; \end{aligned}$$

and this result is true even if the point P lies between the two parallel lines. The following theorem, then, holds:

141. Theorem V.—*The moment of a couple is the product of the common magnitude of the forces and the perpendicular distance between them.*

The moment is positive if the sense of the couple is counterclockwise; negative, if it is clockwise. This is most easily tested by imagining the arm of the couple to be a rigid bar pivoted at its center and the vectors to represent the velocity of its ends. If the bar is rotating counterclockwise the sense of the rotation (and, therefore, also the sense of the couple) is positive. If the sense of the rotation is clockwise the moment of the couple is negative.

142. Theorem VI.—*Two couples which act in the same plane and which have the same moment are equivalent.*

This statement will be proved by showing that if one of the couples is reversed the resulting system of two couples is in equilibrium.

Let Λ_1 and Λ_2 be the lines of action of the forces $\mathbf{F}_1$ and $\mathbf{F}_2$ of the first couple, and L_1 and L_2 be the lines of action of the second

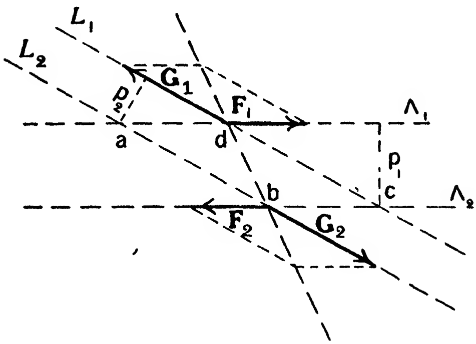


Fig. 64.

couple reversed $\mathbf{G}_1$ and $\mathbf{G}_2$. Slide the forces of the couples along their lines to the points of intersection with the lines of action of the other couple, as represented in Fig. 64. This is possible by the principle of the transmissibility of force. After this is done, $\mathbf{F}_1$ and $\mathbf{G}_1$ act upon the same particle and $\mathbf{F}_2$ and $\mathbf{G}_2$ act on the same particle.

Let

$$\mathbf{R}_1 = \mathbf{F}_1 + \mathbf{G}_1, \qquad \mathbf{R}_2 = \mathbf{F}_2 + \mathbf{G}_2.$$

Then

$$\mathbf{R}_1 + \mathbf{R}_2 = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{G}_1 + \mathbf{G}_2 = 0,$$

which shows that $\mathbf{R}_1$ and $\mathbf{R}_2$ are equal and opposite. It remains to be shown that they act in the same straight line.

Since the couples have the same moments,

$$p_2G_1 = p_1F_1, \qquad \text{or} \qquad \frac{F_1}{G_1} = \frac{p_2}{p_1}.$$

The lines L_1, L_2, Λ_1, and Λ_2 form a parallelogram *abcd* in which

$$\frac{ad}{dc} = \frac{p_2}{p_1} = \frac{F_1}{G_1}.$$

Since the corresponding angles also are equal, it follows that the parallelogram *abcd* is similar to the parallelograms of which $\mathbf{F}_1$, $\mathbf{G}_1$ and $\mathbf{F}_2$, $\mathbf{G}_2$ are the sides. The straight line which contains the diagonal *bd*, therefore, also contains the diagonals of the parallelograms $\mathbf{F}_1\mathbf{G}_1$ and $\mathbf{F}_2\mathbf{G}_2$. Hence, $\mathbf{R}_1$ and $\mathbf{R}_2$ lie in the same straight line and the system of forces is in equilibrium. Q.E.D.

It follows that a couple which is acting in a plane can be represented by any pair of equal and opposite forces in the plane provided the perpendicular distance between their lines of action is such that the product $p \cdot F$ is equal to the moment of the given couple and the sense of rotation is preserved.

143. Theorem VII.—*Two couples which act in parallel planes and which have the same moment are equivalent.*

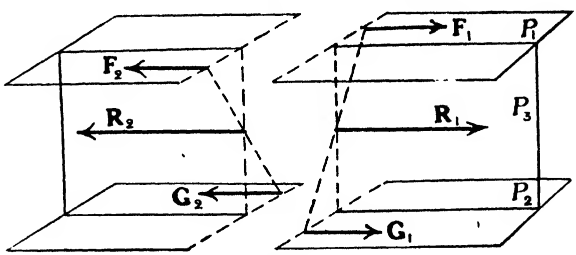


FIG. 65.

As before, one of the couples will be reversed, and it will then be shown that the system of two couples is in equilibrium. Let $\mathbf{F}_1$ and $\mathbf{F}_2$ be the first couple acting in the plane P_1, and $\mathbf{G}_1$ and $\mathbf{G}_2$ be the second couple reversed, acting in the parallel plane P_2. Let P_3 be a plane perpendicular to P_1 and P_2. Let $\mathbf{F}_1$ and $\mathbf{F}_2$ be parallel to the intersection of P_1 and P_3 and at equal distances from it. Likewise, let $\mathbf{G}_1$ and $\mathbf{G}_2$ be parallel to the intersection of P_2 and P_3 and at equal distances from it. Furthermore, let $F_1 = F_2 = G_1 = G_2$, so that the perpendicular distance between $\mathbf{F}_1$ and $\mathbf{F}_2$ is equal to the perpendicular distance between $\mathbf{G}_1$ and $\mathbf{G}_2$.

Let the origin of $\mathbf{F}_1$ and $\mathbf{G}_1$ lie in a plane which is perpendicular to P_1, P_2, and P_3; and also the origin of $\mathbf{F}_2$ and $\mathbf{G}_2$, as in Fig. 65. This arrangement is possible by theorem VI. Then $\mathbf{F}_1$ and $\mathbf{G}_1$ can be replaced by the single force $\mathbf{R}_1$, and $\mathbf{F}_2$ and $\mathbf{G}_2$ can be replaced by the single force $\mathbf{R}_2$. It is evident that

$$\mathbf{R}_1 + \mathbf{R}_2 = 0,$$

and that each of them lies in the plane P_3 and also in a plane parallel to P_1 and P_2 and halfway between them. They, therefore, lie in the same straight line and form a system in equilibrium. Consequently, the two couples are in equilibrium.

144. Geometric Representation of a Couple.—It is interesting to notice that a couple can be represented by a directed segment of a straight line. The *axis of a couple* is any line perpendicular to the plane of the couple. If a segment of the axis of the couple be taken equal in length to the moment of the couple, and its direction along the line be determined according as the moment of the couple is positive or negative, then this directed segment completely represents the couple.

145. Theorem VIII.—*Couples are vectors.*

It has been shown that couples can be represented by directed lines. To show that they are vectors, it is necessary to show that they combine according to the parallelogram law.

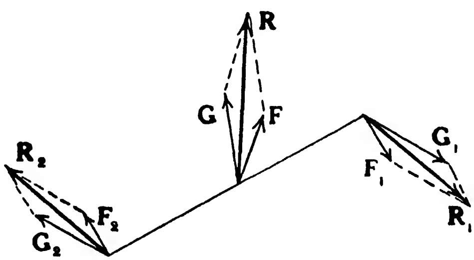


Fig. 66.

Let $\mathbf{F}$ represent one couple and $\mathbf{G}$ another. They can be taken at the same origin, since the axis of a couple is any line perpendicular to the plane of the couple. Let $\mathbf{F}_1$ and $\mathbf{F}_2$ with the arm a be the couple represented by $\mathbf{F}$, and $\mathbf{G}_1$ and $\mathbf{G}_2$ with the same origins and the same arm a be the couple represented by $\mathbf{G}$, (Fig. 66). Then

$$F = aF_1, \qquad G = aG_1.$$

Moreover, since $\mathbf{F}$ is normal to the plane of $\mathbf{F}_1$, $\mathbf{F}_2$ and $\mathbf{G}$ is normal to the plane of $\mathbf{G}_1$, $\mathbf{G}_2$, the angle $\widehat{\mathbf{FG}}$ is the same as the angles $\widehat{\mathbf{F}_1\mathbf{G}_1}$ and $\widehat{\mathbf{F}_2\mathbf{G}_2}$. The corresponding parallelograms, therefore, are similar. If

$$\mathbf{R} = \mathbf{F} + \mathbf{G}, \qquad \mathbf{R}_1 = \mathbf{F}_1 + \mathbf{G}_1, \qquad \mathbf{R}_2 = \mathbf{F}_2 + \mathbf{G}_2,$$

then also

$$\widehat{\mathbf{R}_1\mathbf{G}_1} = \widehat{\mathbf{R}_2\mathbf{G}_2} = \widehat{\mathbf{RG}},$$

and the vectors $\mathbf{R}_1$ and $\mathbf{R}_2$ lie in a plane which is perpendicular to $\mathbf{R}$. Since $F = aF_1$ and $G = aG_1$, it follows that

$$R = aR_1.$$

The sum, therefore, of the two couples $\mathbf{F}_1$, $\mathbf{F}_2$ and $\mathbf{G}_1$, $\mathbf{G}_2$ is the couple $\mathbf{R}_1$, $\mathbf{R}_2$ which is represented by the directed segment $\mathbf{R}$. But since

$$\mathbf{R} = \mathbf{F} + \mathbf{G},$$

it is proved that couples combine according to the parallelogram law and are therefore vectors.

It should be noticed that a couple acting on a rigid body is a *free vector*, that is, it can be shifted about anywhere provided its direction and magnitude are preserved; while a force acting on a rigid body is a *sliding vector*, that is, it can be slid anywhere along its line of action but cannot be taken out of that line.

The resultant of any number of couples acting on a rigid body is again a couple whose geometric representation is the vector sum of the vectors which represent the individual couples. Furthermore, a couple $\mathbf{C}$ can be resolved into components $\mathbf{C}_x$, $\mathbf{C}_y$, and $\mathbf{C}_z$ along three mutually perpendicular axes and C_x, C_y, and C_z are the moments of the forces of $\mathbf{C}$ with respect to the x-, y-, and z-axes, respectively.

The moment of a couple is frequently called a *torque*.

146. Theorem IX.—*If a rigid body, acted upon by a couple, is given a small rotation $d\theta$ about any axis which is parallel to the axis of the couple, the work done by the couple is the product of the angular displacement of the body and the moment of the couple.*

Let B be a rigid body which is free to turn about an axis through O perpendicular to the sheet of the paper. If B is acted upon by a couple of moment M, the axis of which is parallel to the axis through O, the couple can be represented by a pair of forces $\mathbf{F}$ and $-\mathbf{F}$ in the plane of the paper, the points of application of which

lie on a straight line which is perpendicular to their lines of action and which passes through O (Fig. 67).

Let the distances of these points of application from O be b and $a + b$. Then the moment of the couple is

$$M = a \cdot F.$$

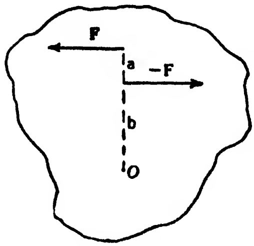


FIG. 67.

Now let the body be turned through an angle $d\theta$. The work done by the force **F** is $F \cdot (a + b) \cdot d\theta$ and the work done by the force $-$**F** is $-F \cdot b \cdot d\theta$. The signs are opposite because the displacement of the point of application of one of the forces is in the direction of the force while the displacement of the point of application of the other force is opposite to the direction of the force. The total work done by the couple, therefore, is

$$W = a \cdot F \cdot d\theta = Md\theta$$

which proves the theorem.

If the axis of the couple is not parallel to the axis of rotation, the couple can be resolved into two couples, the axis of one being parallel to the axis of rotation and the axis of the other being perpendicular to it. The second couple does no work in a rotation about the given axis of rotation, for the forces of the couple lie in a plane which passes through the axis of rotation, and the displacements of the points of application of these forces are perpendicular to the directions of the forces. The above theorem can be applied to the parallel component. Hence, the more general theorem follows:

147. Theorem X.—*If a rigid body, acted upon by a couple, is given a small rotation about any axis whatever, the work done by the couple is equal to the angular displacement of the body multiplied by the component of the moment of the couple which is parallel to the axis of rotation.*

Let **A** denote the infinitesimal rotation and **C** the couple, then the work done by the couple is the scalar product $\mathbf{A} \cdot \mathbf{C}$.

148. Theorem XI.—*If the potential energy of a rigid body is $V(\theta)$, where θ defines the orientation of the body about any fixed line, the moment of the forces acting upon the body with respect to this line is*

$$M = -\frac{\partial V}{\partial \theta}.$$

(The positive end of the axis of rotation is understood to be the direction in which a right-handed nut advances when it is rotating in the sense of θ increasing.)

To prove this theorem, let it be supposed that the rigid body is rotated about the given line through a small angle $d\theta$. Then the increase in the potential energy is

$$\frac{\partial V}{\partial \theta} \cdot d\theta.$$

This measures the amount of work which has been done upon the body. On the other hand, the work which has been done is measured by the product of the angular displacement $d\theta$ and the moment of the couple M_1 which must be applied in order to produce rotation against the forces which are acting naturally upon the body. Hence,

$$M_1 d\theta = \frac{\partial V}{\partial \theta} d\theta, \qquad \text{or} \qquad M_1 = \frac{\partial V}{\partial \theta}.$$

The moment of this couple is exactly opposite to the moment M which the given forces produce; that is, $M_1 = -M$, and therefore,

$$M = -\frac{\partial V}{\partial \theta}.$$

149. Example—the Bifilar Pendulum. A uniform bar, of weight W and length l, is suspended by two light strings of equal length a attached to its two extremities and to two points in the ceiling at a distance l apart, so that when at rest the bar is in a horizontal position. The bar is rotated through an angle θ, the center of the bar rising vertically. What couple is required to hold the bar in this position?

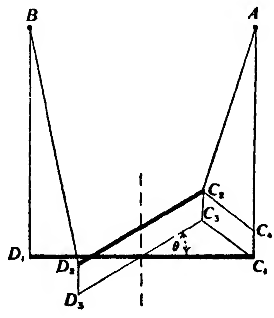


FIG. 68.

In Fig. 68, let

C_1D_1 be the bar in its position of rest;
C_2D_2, the bar after rotation;
C_3D_3, the projection of C_2D_2 on the horizontal plane through C_1D_1;
AC_4, the projection of AC_2 on the vertical AC_1;
$C_1C_4 = h$, the height through which the bar is raised;
M_1, the moment of the required couple.

The potential energy of the bar in its rotated position relative to its position of rest is

$$V = W \cdot h,$$

and, by theorem XI,

$$M_1 = W\frac{dh}{d\theta}.$$

From the diagram, it is seen that

$$\overline{AC_2}^2 - \overline{C_2C_4}^2 = (\overline{AC_1} - \overline{C_1C_4})^2,$$

or
$$a^2 - \overline{C_1C_3}^2 = (a - h)^2.$$

But
$$\overline{C_1C_3} = l \sin \frac{1}{2}\theta,$$

and therefore
$$h = a - \sqrt{a^2 - l^2 \sin^2 \frac{1}{2}\theta},$$

from which is derived

$$M_1 = -\frac{Wl^2 \sin \theta}{4\sqrt{a^2 - l^2 \sin^2 \frac{1}{2}\theta}},$$

which is the moment of the couple required to hold the bar in this position. The action of gravity and the constraint of the strings is a couple acting upon the bar, the moment of which is

$$M = -M_1.$$

150. Theorem XII.—*Any system of forces acting upon a rigid body can be replaced by a suitably chosen single force acting at an arbitrarily chosen point and a couple.*

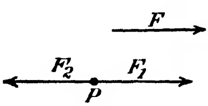


Fig. 69.

Let P be the point selected, and let **F** be any force whose line of action does not pass through P. Let two forces $\mathbf{F}_1$ and $\mathbf{F}_2$, such that

$$\mathbf{F} + \mathbf{F}_2 = 0, \qquad \mathbf{F}_1 + \mathbf{F}_2 = 0, \qquad F = F_1 = F_2,$$

be introduced at P. The force **F** is now replaced by a force $\mathbf{F}_1$ acting at P and a couple, **F** and $\mathbf{F}_2$. Let this be done for every force whose line of action does not already pass through P. The forces now acting at P have a unique resultant **R** which also acts at P; and since the couples are free vectors they also have a unique resultant couple **C** (Sec. 145).

If the point P is the center of gravity and if the rigid body is free to move, the resultant force **R** produces a pure translation and the couple **C** produces a pure rotation about some axis which passes through the center of gravity.

151. Theorem XIII.—*Any system of forces acting upon a rigid body can be replaced by a wrench.*

A *wrench* is a system of forces composed of a single force and a couple, in which the axis of the couple is parallel to the line of the force.

Let the given system of forces be replaced by an equivalent system, *viz.*, a force **F** and a couple **C** the axis of which makes an angle α with the direction of **F**. Let **C** be resolved into two couples $\mathbf{C}_1$ and $\mathbf{C}_2$ with $\mathbf{C}_1$ parallel to **F** and $\mathbf{C}_2$ perpendicular to it, their moments being, respectively,

$$C_1 = C \cos \alpha, \qquad C_2 = C \sin \alpha.$$

The couple $\mathbf{C}_2$ can be regarded as a pair of forces $\mathbf{F}_1$ and $\mathbf{F}_2$ lying in a plane which is perpendicular to $\mathbf{C}_2$ and which contains **F**. Let the arm a of the couple be chosen so that

$$F_1 = F, \qquad a = \frac{C_2}{F};$$

and, furthermore, let the line of $\mathbf{F}_1$ coincide with the line of **F** and its direction be opposite. Then

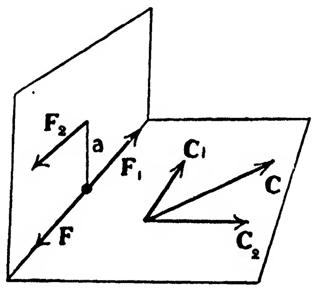


FIG. 70.

$$\mathbf{F} + \mathbf{F}_1 = 0,$$

and there remains, in the place of the original system, a force $\mathbf{F}_2$, which is equal in magnitude to **F** and parallel to it, and a couple $\mathbf{C}_1$, the axis of which is parallel to **F** and therefore parallel to $\mathbf{F}_2$; that is to say, a wrench remains. Hence, any system of forces which act upon a rigid body can be replaced by an equivalent wrench.

152. Theorem XIV.—*Any system of forces acting upon a rigid body, which is not equivalent to a single couple, can be replaced by two forces which are perpendicular to each other, but whose lines of action, in general, do not intersect.*

Let the given system of forces be replaced by a single force **F** acting at P, and a couple **C**. Let P_2 be the plane which contains **F** and which is parallel to **C**, or contains **C**. Let P_1 be the plane through P which is perpendicular to **C**. The force **F** can be resolved into two forces acting at P, one of which $\mathbf{F}_2$ is perpendicular to P_1, and the other $\mathbf{F}_1$ lies in the intersection of P_1 and P_2.

The couple **C** can be represented by two forces $\mathbf{G}_1$ and $\mathbf{G}_2$ lying in the plane P_1 with the arm a, and such that

$$aG_1 = aG_2 = C.$$

Let
$$\mathbf{G}_2 = -\mathbf{F}_1$$

be applied at P. Then $\mathbf{G}_2$ and $\mathbf{F}_1$ are in equilibrium by themselves; and there remains $\mathbf{F}_2$, acting at P and perpendicular to P_1, and $\mathbf{G}_1$ acting in the plane P_1 at a distance a from P. It is evident that $\mathbf{G}_1$ and $\mathbf{F}_2$ are mutually perpendicular and, if $\widehat{\mathbf{FF}_2} = \theta$,

$$a = \frac{C}{F \sin \theta}.$$

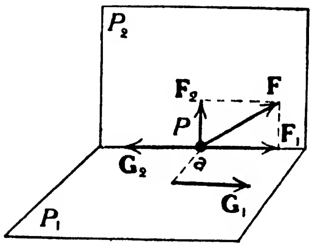


FIG. 71.

The point P is arbitrary provided it does not lie upon the axis of the equivalent wrench.

153. General Conditions for the Equilibrium of a Rigid Body. It is now possible to state the conditions that must exist in order that a rigid body may be in equilibrium under a given system of forces $\mathbf{F}_1, \mathbf{F}_2, \cdots, \mathbf{F}_n$.

154. Theorem XV.—*Necessary and sufficient conditions that a rigid body, acted upon by the forces* $\mathbf{F}_1, \mathbf{F}_2, \cdots \mathbf{F}_n$ *at the points* $p_1, p_2, \cdots p_n$, *shall be in equilibrium are (a) that the sum of the projections of the forces upon any three mutually perpendicular axes is zero, and (b) that the sum of the moments of the forces with respect to these same three axes is zero.*

Let the system of forces $\mathbf{F}_1, \mathbf{F}_2, \cdots, \mathbf{F}_n$ be replaced by the equivalent system: the force $\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 + \cdots + \mathbf{F}_n$ acting at the center of gravity and the couple **C**. The effect of **F** acting at the center of gravity is to produce a translation of the body. In order that a rigid body, initially at rest, may not be set into translation, it is necessary and sufficient that $\mathbf{F} = 0$. This compels the three conditions

$$\mathbf{F}_x = \mathbf{F}_y = \mathbf{F}_z = 0.$$

The effect of the couple **C** is to produce a rotation of the body about some axis through the center of gravity. In order, therefore, that the body may not rotate, it is necessary and sufficient that $\mathbf{C} = 0$, or, what is equivalent, that its three components

$$\mathbf{C}_x = \mathbf{C}_y = \mathbf{C}_z = 0;$$

that is to say, the sum of the moments of the forces with respect to each of the three axes is zero. It is evident that if these conditions are satisfied, the sum of the moments with respect to any axis whatever is zero.

155. Equilibrium of Coplanar Forces.—If the system of forces is coplanar, let their common plane be taken as the xy-plane. Then the conditions

$$\mathbf{F}_z = 0, \qquad \mathbf{C}_x = 0, \qquad \mathbf{C}_y = 0$$

are necessarily satisfied; the first, because the z-component of each force is zero, and the other two because each force either intersects both of the x- and y-axes, or intersects one of them and is parallel to the other. There remain only three conditions to be satisfied, namely,

$$\mathbf{F}_x = 0, \qquad \mathbf{F}_y = 0, \qquad \mathbf{C}_z = 0,$$

or stated in another form:

156. Theorem XVI.—*A system of coplanar forces is in equilibrium if the sum of the moments of all of the forces with respect to each of three non-collinear points is zero.*

The given system of forces is either equivalent to a single force (Sec. 127) with a definite line of action, or to a couple. If it is a couple, it is sufficient that the moment vanishes at any one point, since it must vanish then at all points. If it is a single force, its moment cannot vanish at three non-collinear points unless the force itself vanishes. Its moment might vanish at two specified points, however, for the two points might lie on its line of action.

A very useful theorem for a rigid body which is acted upon by three forces only is the following:

157. Theorem XVII.—*If a rigid body is in equilibrium under the action of three coplanar forces, the three lines of action meet in a point and the forces satisfy Lami's theorem.*

The force $\mathbf{F}_1$ acting at p_1 and the force $\mathbf{F}_2$ acting at p_2 are equivalent to a force

$$\mathbf{R} = \mathbf{F}_1 + \mathbf{F}_2$$

acting at p_3, the point of intersection of their lines of action. The rigid body is now in equilibrium under the action of two forces $\mathbf{R}$ and $\mathbf{F}_3$. Therefore, $\mathbf{R} = -\mathbf{F}_3$ and they both lie in the

same straight line and the line of action of $\mathbf{F}_3$ also passes through p_3. Consequently,

$$\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 = 0,$$

and the three lines of action meet at a point. If each of the three forces be regarded as acting at p_3, it is evident at once that Lami's theorem must be satisfied. It will be left to the student to prove that if a body is in equilibrium under the action of three forces, the forces are necessarily coplanar.

158. A Suspended Rod.—A heavy uniform rod of length $2l$ is supported at its extremities by two strings of lengths l_1 and l_2 which are tied to the same point of the ceiling. Find the tension in the strings.

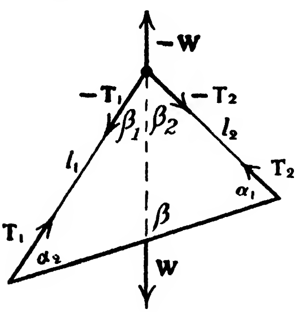


FIG. 72.

The rod is acted upon by three forces in the same plane, *viz.*, $\mathbf{T}_1$, $\mathbf{T}_2$, and $\mathbf{W}$. Since the three lines of action meet in a point and the two tensions meet at the point of suspension, the line of $\mathbf{W}$, the weight of the rod, also passes through the point of suspension. The center of the rod, therefore, is directly below this point.

Let α be the angle between the two strings and α_1 and α_2 the angles which the strings make with the rod. Let the vertical line through the point of suspension divide the angle α into two parts β_1 and β_2 and make an angle β with the rod. Then the three forces which pass through the point of suspension are in equilibrium and, by Lami's theorem,

$$\frac{T_1}{\sin \beta_2} = \frac{T_2}{\sin \beta_1} = \frac{W}{\sin \alpha}.$$

From the geometry of the two smaller triangles,

$$\sin \beta_2 = \frac{l}{l_2} \sin \beta, \qquad \sin \beta_1 = \frac{l}{l_1} \sin \beta.$$

From these relations, it follows that

$$\frac{T_1}{l_1} = \frac{T_2}{l_2} = \frac{Wl}{l_1 l_2} \cdot \frac{\sin \beta}{\sin \alpha} = \frac{W}{\sqrt{2l_1^2 + 2l_2^2 - 4l^2}}$$

and the two tensions are proportional to the lengths of the strings.

159. A Man Stands on a Ladder.—Two uniform ladders, each of length 12 feet and weighing 20 pounds, freely jointed at the top, are connected at the bottom by a rope 5 feet in length and rest on smooth ice. A man weighing 160 pounds stands on one of the ladders 9 feet from its foot. What is the tension in the rope?

Let AC and BC be the two ladders, and let M be the point at which the man is standing. Since the ice is smooth, its actions upon the ladders $\mathbf{P}_A$ and $\mathbf{P}_B$ are vertical; and their sum is equal to the weight of the entire system. Therefore,

$$P_A + P_B = 200 \text{ pounds}.$$

Regarding the entire system ABC as a rigid system, it is seen that the sum of the moments of the forces acting upon it with respect to the point A is zero. Hence,

$$5P_B = 20\left(\frac{5}{4} + \frac{15}{4}\right) + \left(160 \times \frac{15}{8}\right) = 400,$$

and therefore $\quad P_A = 120, \qquad P_B = 80.$

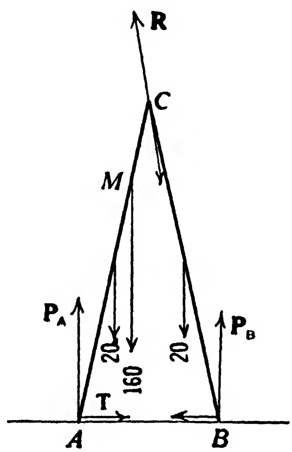


FIG. 73.

Now consider the moments of the forces which are acting upon the ladder AC alone with respect to the point C. There is a pressure of the ladder BC at the point C whose magnitude and direction is unknown, but as it passes through C its moment is zero. From the moments of the remaining forces, there is obtained

$$\frac{5}{2}P_A = T\sqrt{144 - \frac{25}{4}} + \left(20 \times \frac{5}{4}\right) + \left(160 \times \frac{5}{8}\right),$$

and therefore

$$T = \frac{175}{\sqrt{137.75}} = 14.91 \text{ approximately}.$$

The magnitude and direction of the force which the ladder BC impresses upon the ladder AC at the point C can now be found. Let this force be denoted by $\mathbf{R}$, and let the magnitudes of its vertical and horizontal components be R_v and R_h. Since the sum of the components of all of the forces acting upon AC with respect to any axis whatever is zero, it is found, resolving vertically and horizontally, that

$$R_v = 60 \text{ pounds}, \qquad R_h = T = 14.91 \text{ pounds}.$$

There is, of course, a force $-\mathbf{R}$ acting at C upon the ladder BC. It will be found upon examination that the forces acting on the ladder BC also are in equilibrium.

160. Four Spheres in a Bowl.—Four equal spheres rest in contact at the bottom of a spherical bowl, their centers being at the corners of a horizontal square. A fifth sphere of the same radius and density is placed upon them. It is required to find under what conditions equilibrium exists, all contacts being regarded as smooth.

Let the radius of the bowl be a and the radius of the spheres be b. If equilibrium exists, each sphere is in equilibrium under the

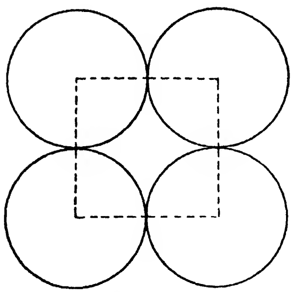


FIG. 74.

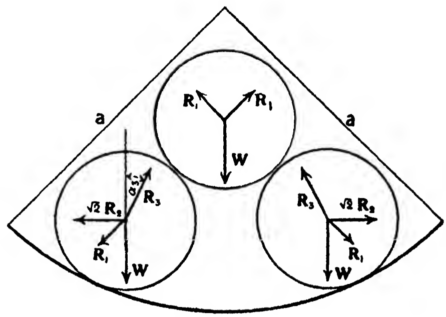


FIG. 75.

action of its weight and the reactions of the bowl and the other spheres. Since the contacts are smooth, all of the forces which act upon a given sphere pass through the center of the sphere. In particular, the top ball is in equilibrium under the action of its own weight and the reactions of the four lower spheres. It is assumed from symmetry that these four reactions are numerically equal. Each of them makes an angle of 45° with the vertical. If R_1 is the magnitude of each of these reactions and W is the weight of each sphere, then

$$4R_1 \cos 45^\circ = W;$$

whence
$$R_1 = \frac{W}{2\sqrt{2}}.$$

Now consider the actions of the lower spheres on one another. Again, from considerations of symmetry, these actions are all equal and will be denoted by R_2. Since they pass through the centers of the spheres, the two actions on each sphere can be

combined into a single horizontal force the magnitude of which is $\sqrt{2}R_2$, and the direction is perpendicular to the axis of the bowl. Let the magnitude of the action of the bowl on each sphere be denoted by R_3, since they are all of the same magnitude. If the angle which this action makes with the vertical is α_3, then

$$\sin \alpha_3 = \frac{\sqrt{2}b}{a - b}.$$

Thus the four forces which act upon each sphere can be regarded as acting at the center of the sphere; they lie in the same plane; their directions are known; and the magnitude of two of the forces R_1 and W are known. On resolving these forces horizontally and vertically, the following equations are obtained:

$$\sqrt{2}R_2 = R_3 \sin \alpha_3 - R_1 \sin 45°,$$
$$R_3 \cos \alpha_3 = R_1 \cos 45° + W,$$

whence

$$R_3 = \frac{5}{4} W \sec \alpha_3$$

and

$$R_2 = \frac{W}{4\sqrt{2}} (5 \tan \alpha_3 - 1).$$

The pressures between the four lower spheres vanish if $R_2 = 0$; that is, if $\tan \alpha_3 = 1/5$. Equilibrium can exist only if $\tan \alpha_3 \geqq 1/5$ and, in order that this may be true, it is necessary that

$$a \leqq (2\sqrt{13} + 1)b.$$

161. A Heavy Cube on a Rough Inclined Plane.—A heavy cube, of edge $2a$, is placed on a rough inclined plane with an inclination of α, the lower edge being horizontal. Determine the conditions of equilibrium, the angle of friction being ϵ. It will be supposed that the angle of friction is greater than the angle of inclination and that the cube is at rest on the inclined plane. The cube is acted upon by two forces only, the weight **W** acting in a vertical line through its center, and the reaction of the plane. These two forces, therefore, lie in the same straight line and are equal and opposite. As the inclined plane acts only on the base of the cube, it will be of interest to analyze this reaction.

For perfectly rigid bodies, this problem, like that of the table with four legs, is indeterminate. If it is supposed, however, that the cube makes a slight impression upon the plane, then, by a law similar to Hooke's law (Sec. 208), the pressure on any small

area of the base will be proportional to the depth of the impression. With uniform substances the boundary between the plane and the cube is still a plane but not necessarily parallel to the inclined plane. In Fig. 77, let Q be the horizontal line in which these two planes intersect. The depth of the impression will then be proportional to the distance from the line Q and, therefore, the pressure will be constant along a strip of width dx parallel to the line Q or the lower edge of the cube. Let x be the

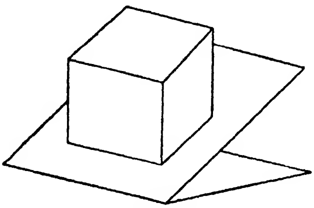


FIG. 76.

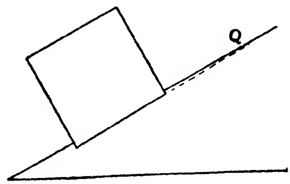


FIG. 77.

distance up the inclined plane measured from the lower edge of the cube. As the depth of the impression is a linear functon of x, so also is the pressure which is acting vertically (pressure = force per unit area). Let W be the weight of the cube. Then

$$dW = k(1 - hx)dx \tag{1}$$

is the weight carried by a strip of length $2a$ and of width dx across the base of the cube, h and k being two constants which are to be determined by the conditions of equilibrium. One of these conditions is that the total pressure on the base is equal to the weight of the cube, that is,

$$W = \int_0^{2a} k(1 - hx)dx. \tag{2}$$

The other is that the center of gravity of the pressures must be at the point where the line of action of $\mathbf{W}$ pierces the base of the cube, which is at the distance $a(1 - \tan\alpha)$ from the lower edge. Hence,

$$\frac{\int_0^{2a} k(1 - hx)xdx}{\int_0^{2a} k(1 - hx)dx} = a(1 - \tan\alpha). \tag{3}$$

The integration of Eqs. (2) and (3) gives the two equations

$$2ka(1 - ha) = W,$$

and

$$ha = \frac{3\tan\alpha}{1 + 3\tan\alpha}.$$

Therefore,

$$k = \frac{W}{2a}(1 + 3 \tan \alpha).$$

On substituting these values of h and k in Eq. (1), it is found that the weight carried by an elemental strip is

$$dW = \frac{W}{2a}\left[1 + 3\left(1 - \frac{x}{a}\right) \tan \alpha\right] dx;$$

and this expression divided by the area of the strip $2adx$ gives the pressure at the distance x from the lower edge, namely,

$$P_x = \frac{W}{4a^2}\left[1 + 3\left(1 - \frac{x}{a}\right) \tan \alpha\right].$$

At the upper edge, $x = 2a$, the pressure becomes

$$P_{2a} = \frac{W}{4a^2}[1 - 3 \tan \alpha],$$

which vanishes if $\tan \alpha = 1/3$, or $\alpha = 18° 26'$. Evidently negative pressures are not admissible, and therefore these formulas hold only if $\tan \alpha \leqq 1/3$. For this value of α, the point Q lies on the upper edge of the base of the cube, and for larger values of α it is below the upper edge, so that the upper edge of the base of the cube does not touch the inclined plane. The limits of integration must be altered to take this fact into account. The formula for the pressure (Eq. (1)) changes from positive to negative for $x = 1/h$ and, therefore, if $\alpha > 18° 26'$, this must be the upper limit of integration instead of $x = 2a$. On repeating the calculation for this limit and denoting the change by the subscript 1, it is found that

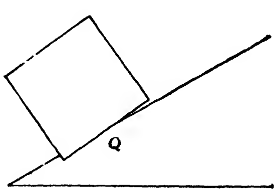


FIG. 78.

$$h_1 a = \frac{1}{3(1 - \tan \alpha)}, \qquad k_1 = \frac{2W}{3a(1 - \tan \alpha)},$$

$$P_1 = \frac{W}{3a^2(1 - \tan \alpha)}\left[1 - \frac{x}{3a(1 - \tan \alpha)}\right],$$

$$x \leqq \frac{3}{2}a(1 - \tan \alpha), \qquad \alpha \geqq 18° 26'.$$

The expressions for the pressure at the lower edge of the base are

$$P_0 = \frac{W}{4a^2}(1 + 3\tan\alpha), \qquad \tan\alpha \leqq \frac{1}{3},$$

$$P_0 = \frac{W}{3a^2(1 - \tan\alpha)}, \qquad 1 > \tan\alpha \geqq \frac{1}{3},$$

which show how the pressures increase with α. At $\alpha = 45°$, the entire weight of the cube rests on the lower edge and the pressure becomes infinite. The assumption that the cube is essentially rigid, therefore, fails before this point is reached.

If α is increased beyond 45°, the line of **W** falls outside of the base of the cube. The reaction of the plane cannot pass the lower edge. Hence, the moments of the forces are no longer zero and the cube topples over.

In the above analysis, it has been assumed that the angle of friction was always greater than α; but if $\epsilon < 45°$ there is a value of α which is equal to ϵ. For this value $\alpha = \epsilon$, equilibrium exists. The two forces **W** and **R** lie in the vertical line which passes through the center of the cube and are equal and opposite. But if α is increased slightly beyond ϵ, the frictional component of **R** is no longer able to balance the component of **W** parallel to the plane, although the normal component continues to balance the normal component of **W**. The reaction **R** continues to pass through the center of the cube but it is no longer in the vertical; **R** + **W** applied at the center of gravity is equivalent to a single force parallel to the plane, and the cube starts to slide down the plane.

162. Forces Acting on a Cube.—Given a cube of which the edges, of length two feet, are horizontal and vertical. Let the corners of the base be numbered 1, 2, 3, and 4, in order; and let the corresponding corners of the top be 5, 6, 7, and 8. A force is acting upon the cube at each corner, and the projections of these forces upon the axes of a trihedron with origin at the center of the cube and axes parallel to the edges of the cube, expressed in pounds, is as follows:

Corner	x	y	z	Corner	x	y	z
1	0	0	0	5	0	1	1
2	0	0	1	6	1	0	1
3	0	1	0	7	1	1	0
4	1	0	0	8	1	1	1

Determine the equivalent force and couple for the center of gravity of the cube.

The sum of the components of all of the forces acting along each of the x-, y-, and z-axes is $+4$. Hence, the equivalent force **F** acting at the center of gravity is equally inclined to each of the three axes and its magnitude is

$$F = 4\sqrt{3} \text{ pounds.}$$

Let the trihedron be right handed, as drawn in the figure. In a left-handed trihedron, the positive x- and y-axes are interchanged. A positive rotation about the z-axis makes the positive x-axis move toward the positive y-axis. A positive rotation

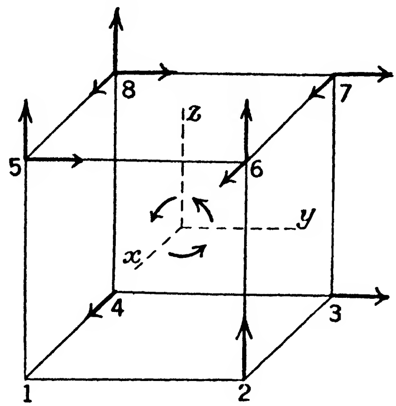


FIG. 79.

about the x-axis makes the positive y-axis move toward the positive z-axis. A positive rotation about the y-axis makes the positive z-axis move toward the positive x-axis. The rotations are therefore cyclical in the order

$$x \longrightarrow y \longrightarrow z \longrightarrow x.$$

Or, if an ordinary right-handed nut is thought of as turning on a threaded axis, in every case a positive rotation is such as would make the nut advance toward the positive end of the axis; and this convention holds for any axis whatever.

With this understanding as to the meaning of positive and negative rotations, the moments of each of the components of the forces can be computed with respect to the x-, y-, and z-axes, as follows:

Corner	x	y	z	Corner	x	y	z
1	0	0	0	6	0	+1	−1
2	+1	−1	0		+1	−1	0
3	+1	0	−1	7	0	+1	−1
4	0	−1	+1		−1	0	−1
5	−1	0	+1	8	0	+1	+1
	−1	−1	0		−1	0	−1
					−1	+1	0
				Total	−2	0	−2

The components of the couple **C** are, therefore,

$$\mathbf{C}_x = -2, \qquad \mathbf{C}_y = 0, \qquad \mathbf{C}_z = -2.$$

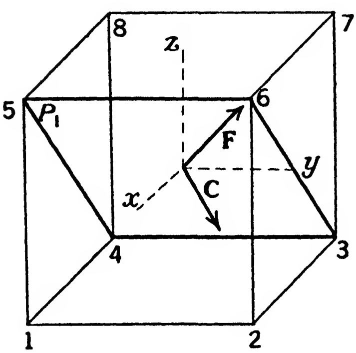


Fig. 80.

The vector **C** itself bisects the angle between the negative ends of the x- and z-axes, and its magnitude is $2\sqrt{2}$ foot-pounds. Both **C** and **F** lie in the plane P_1 which passes through the edges 3-4 and 5-6, the equation of which is $x - z = 0$.

163. The Equivalent Wrench. If the vector **C** is resolved along the line of **F** and perpendicular to it, it follows that

$$C_1 = C \cos \widehat{\mathbf{CF}}, \qquad C_2 = C \sin \widehat{\mathbf{CF}}.$$

If θ is the angle between two lines whose direction cosines are $\alpha_1, \beta_1, \gamma_1$ and $\alpha_2, \beta_2, \gamma_2$, then

$$\cos \theta = \alpha_1\alpha_2 + \beta_1\beta_2 + \gamma_1\gamma_2.$$

The direction cosines of **F** and **C** are, respectively,

$$\frac{1}{\sqrt{3}}, \quad \frac{1}{\sqrt{3}}, \quad \frac{1}{\sqrt{3}}; \qquad -\frac{1}{\sqrt{2}}, \quad 0, \quad \frac{-1}{\sqrt{2}}.$$

Hence,

$$\cos \widehat{\mathbf{CF}} = -\sqrt{\frac{2}{3}}, \qquad \sin \widehat{\mathbf{CF}} = +\sqrt{\frac{1}{3}};$$

and since $\quad C = 2\sqrt{2},$

it follows that $\quad C_1 = -\dfrac{4}{\sqrt{3}}, \qquad C_2 = 2\sqrt{\dfrac{2}{3}}.$

The vector $\mathbf{C}_2$, which is perpendicular to $\mathbf{F}$, lies not only in the plane P_1, for which the equation is

$$x - z = 0, \qquad (P_1)$$

but also in the plane P_2 which is perpendicular to $\mathbf{F}$, for which the equation is

$$x + y + z = 0, \qquad (P_2).$$

(In any plane the coefficients of x, y, and z are proportional to the direction cosines of the normal, and if the plane passes through the origin the constant term is zero. Since the direction cosines of $\mathbf{F}$ are all equal, so also are the coefficients of x, y, and z in the equation of the plane perpendicular to $\mathbf{F}$ all equal.) It, therefore, lies in the intersection of the planes P_1 and P_2 and its direction cosines are[1]

$$\frac{-1}{\sqrt{6}}, \quad \sqrt{\frac{2}{3}}, \quad -\frac{1}{\sqrt{6}}.$$

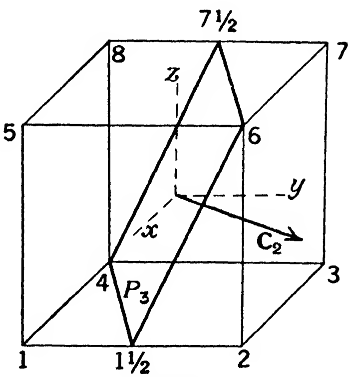


FIG. 81.

Accordingly the plane P_3 through the origin which is perpendicular to $\mathbf{C}_2$ is

$$x - 2y + z = 0, \qquad (P_3)$$

and this plane contains the vector $\mathbf{F}$. It cuts the edges of the cube in the points

$$1\frac{1}{2}, \quad 6, \quad 7\frac{1}{2}, \quad \text{and} \quad 4.$$

Now let the couple $\mathbf{C}_2$ be represented by two forces $\mathbf{F}_1$ and $\mathbf{F}_2$ in the plane P_3, and let

$$F_1 = F_2 = F.$$

Since the moment of $\mathbf{C}_2$ is $C_2 = 2\sqrt{2/3}$, and since $F = 4\sqrt{3}$, the arm of the couple is

$$\frac{C_2}{F} = a = \frac{\sqrt{2}}{6},$$

and this is the perpendicular distance between $\mathbf{F}_1$ and $\mathbf{F}_2$.

Let $\mathbf{F}_1$ lie in the line of $\mathbf{F}$ and be oppositely directed. Then

$$\mathbf{F}_1 + \mathbf{F} = 0,$$

and this pair of forces has no effect on the cube. The force $\mathbf{F}_2$, which lies in the plane P_3, is parallel to $\mathbf{F}$ at the distance $a =$

[1] See, for example, SMITH and GALE, "Analytic Geometry," p. 365.

$\sqrt{2}/6$, and it has the same direction. Its direction cosines are, therefore,

$$\frac{1}{\sqrt{3}}, \quad \frac{1}{\sqrt{3}}, \quad \frac{1}{\sqrt{3}} \qquad (\mathbf{F}_2).$$

The plane P_3 intersects the xy-plane in the line l the equation of which is

$$x - 2y = 0 \qquad (l).$$

The direction cosines of the line l are, therefore,

$$\frac{2}{\sqrt{5}}, \quad \frac{1}{\sqrt{5}}, \quad 0 \qquad (l).$$

Let ω be the angle between $\mathbf{F}_2$ and l. Then

$$\cos \omega = \sqrt{\frac{3}{5}}, \qquad \sin \omega = \sqrt{\frac{2}{5}}.$$

Consider the right triangle formed by the line l, the line $\mathbf{F}_2$, and the perpendicular from the origin to the line of $\mathbf{F}_2$. Let d be the hypothenuse of this triangle. Then

$$d = \frac{a}{\sin \omega} = \frac{\sqrt{5}}{6}.$$

Hence, the line of $\mathbf{F}_2$ pierces the xy-plane at a distance $\sqrt{5}/6$ from the origin, and the coordinates of this point are

$$x_0 = \frac{1}{3}, \qquad y_0 = \frac{1}{6}, \qquad z_0 = 0.$$

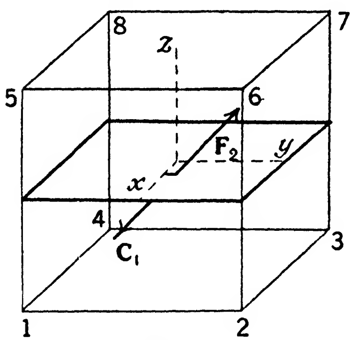


Fig. 82.

The vector $\mathbf{F}_2$, which is the axis of the wrench, is parallel to the diagonal 4-6. The vector $\mathbf{C}_1$ is parallel to $\mathbf{F}_2$ but oppositely directed. The magnitude of the force is $4\sqrt{3}$ pounds, and the moment of the couple, or torque, is $\frac{4}{\sqrt{3}}$ foot-pounds.

Problems XI

1. A light ladder stands upon a rough floor and leans against a smooth wall, making an angle of 60° with the floor. The ladder begins to slip when a man, ascending it, reaches the halfway point. Find the coefficient of friction of the ladder with the floor. *Ans.* $\mu = \sqrt{3}/6$.

2. A heavy uniform rod rests with its extremities on the interior of a rough hemispherical bowl and subtends an angle of 2α at the center of the sphere. Find its maximum inclination to the horizontal. *Ans.*

$$\tan i = \frac{\sin 2\epsilon}{\cos 2\epsilon + \cos 2\alpha}.$$

3. A parabolic curve is placed in a vertical plane with its axis vertical and its vertex downward. Its latus rectum is $4l$. A smooth heavy beam of length b rests against the concave arc and a peg at the focus. Required: the inclination of the beam to the vertical. *Ans.* $i = 2\cos^{-1}(l/b)^{\frac{1}{3}}$.

4. Find the equation of the curve in a vertical plane such that a heavy uniform bar of length $2l$ resting upon the concave side of the curve and upon a peg at the origin is in equilibrium in all positions. *Ans.* $r = l + k \sec\theta$, where θ is the inclination of the bar to the vertical and k is arbitrary.

5. A uniform ladder stands upon a floor where the coefficient of friction is $1/2$ and leans against a wall where the coefficient of friction is $1/3$. If a man, whose weight is three times that of the ladder, ascends the ladder and succeeds in reaching the top, what is the smallest possible inclination of the ladder to the horizontal? *Ans.* $i = \tan^{-1} 41/24$.

6. A rectangular beam, the sides of which are a and b, is placed horizontally with one edge on a rough floor and a second edge against a smooth wall. Prove that equilibrium is not possible unless the height of the highest edge of the beam exceeds

$$\frac{a^2 + 2\mu ab + b^2}{\sqrt{a^2 + (2\mu a + b)^2}}.$$

7. Two rough cylinders lie in contact with each other on a rough horizontal plane. A third similar cylinder is placed on top of them with its axis parallel to the other two axes. If equilibrium exists, show that the coefficient of friction exceeds $2 - \sqrt{3}$.

8. Two heavy uniform rods have their ends connected by two light strings and the whole system is supported at the middle point of one of the rods. Prove that in equilibrium either the rods or the strings are parallel.

9. Two equal uniform spheres, each of weight w and radius a rest in a smooth hemispherical bowl of radius b. Find the pressure P_1, between the two spheres, and also the pressure P_2, of each on the bowl.

10. A circular ring of weight w_1 has a bead of weight w_2 fixed on it. It hangs on a rough peg. What is the smallest angle of friction for which the ring will not slip no matter what point of the ring is in contact with the peg? *Ans.*

$$\epsilon = \sin^{-1}\left(\frac{w_2}{w_1 + w_2}\right).$$

11. Three equal spheres are held together on a horizontal plane by a rubber band around them. If a cube of weight w is placed upon them with one diagonal vertical and the three lower faces of the cube in contact with the spheres, and if all of the contacts are smooth, show that the tension of the band is increased by $1/3 \cdot \sqrt{2/3} \cdot w$.

12. Show that any system of coplanar forces acting upon a rigid body is equivalent to two forces $\mathbf{F}_1$ and $\mathbf{F}_2$ acting at two specified points P_1 and P_2, but that the determination of $\mathbf{F}_1$ and $\mathbf{F}_2$ is not unique.

13. A uniform bar is bent into the form of a V with equal arms and hangs freely from one end. Prove that a plumb line suspended from this end will cross the lower arm at a distance of one-third of the length of the arm from the angle.

14. A uniform ladder of length l and weight w rests on a rough floor and leans against a smooth wall, the inclination to the vertical being α. A force $\mathbf{F}$ is applied horizontally to the ladder at a distance d from the foot of the ladder so as to move the ladder nearer the wall. Show that F must exceed

$$\frac{wl}{l-d}\left(\mu + \frac{1}{2}\tan\alpha\right).$$

15. In the above problem, if $\mathbf{F}$ is horizontal but directed away from the wall, the foot of the ladder will not slip if

$$d > \frac{l\tan\alpha}{2\mu}.$$

IV. VIRTUAL WORK

164. The Degrees of Freedom of a Rigid Body.—A body, or even a particle, is said to be *constrained* if it is not perfectly free to take all of the displacements which are geometrically possible. A *free* particle has three degrees of freedom since three parameters x, y, and z are required to specify its position. A free rigid body has six degrees of freedom, since six parameters, or coordinates, are required to specify its position in space, *viz.*, three coordinates to locate its center of gravity, two coordinates to indicate the direction of a given line which passes through the center of gravity and is fixed in the body, and one coordinate to indicate rotation about the given line, the last three coordinates being angles. A small change in any one of these six coordinates represents a displacement of the body; hence, the six degrees of freedom.

If the center of gravity of the body is compelled to remain on a fixed surface, but the body is otherwise free, there is one degree of constraint

$$f(\bar{x}, \bar{y}, \bar{z},) = 0,$$

and five degrees of freedom. If the center of gravity is compelled to remain on a fixed line, there are two degrees of constraint

$$f_1(\bar{x}, \bar{y}, \bar{z}) = 0, \qquad f_2(\bar{x}, \bar{y}, \bar{z}) = 0,$$

and four degrees of freedom; and so on. *The number of degrees of freedom is the number of parameters necessary to specify completely its geometrical position.* A rough cylinder rolling on a plane has only one degree of freedom since its position is specified by a single parameter, *viz.*, the angle through which it has rolled.

165. Virtual Work.—Although no displacements occur when a body or a system of bodies is in equilibrium, nevertheless the principle of work is of great value in determining the conditions of equilibrium. One can imagine that displacements occur and the imaginary work done by these imaginary displacements is called *virtual work.* There is nothing essentially different from real work about it.

Imagine a body or system of bodies in equilibrium. It may be constrained, or it may not be. The sum of all the forces acting upon it is zero, and the moments of all of the forces are zero, on account of the equilibrium. Suppose the body is given any displacement whatever, which is infinitely small and which is compatible with the constraints. The work required is obviously zero since nothing opposes the displacement. Conversely, if a body or system of bodies is in such a configuration that the work done in *every* infinitesimal displacement which is compatible with the constraints is zero, then the configuration is one of equilibrium.

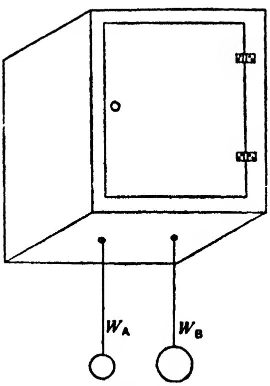


Fig. 83.

166. Concealed Mechanisms.—Suppose, for example, there is a stationary box which conceals a mechanism within. The only information available with respect to it is derived from two ropes A and B (Fig. 83) which hang down from it. It is found that if A moves up or down one inch B freely goes down or up n inches, so that there is no work done within the box. A weight W_B is attached to B. What weight attached to A will leave the system in equilibrium? The principles of forces and moments cannot be applied as in previous examples on account of ignorance of the mechanism; but the principle of work can be applied. Suppose that W_A balances the weight W_B, and then give the system a slight displacement, so that W_A moves upward a distance d and W_B moves downward a distance nd. The work done by W_A

against gravity is $d \cdot W_A$, and the work done by W_B against gravity is $-ndW_B$. Since the total work done vanishes, it is necessary that

$$d \cdot W_A - nd \cdot W_B = 0,$$

whence

$$W_A = nW_B.$$

On opening the box, it may be found that the ropes are attached to the two arms of a lever which have the ratio $1:n$, in which case it is seen by the principle of moments that the solution is correct. Or, there may be found a system of frictionless pulleys in which n strings support the weight W_A and only one supports the weight W_B; or, any one of many other possibilities.

167. The Screw.—The *screw* is a cylinder, usually of wood or metal, on which a uniform thread has been cut in the form of a helix. The screw turns in a hollow cylinder on which a corresponding thread has been cut, and which is called a *nut*. The nut just fits the screw and, as it is turned, it advances along the axis of the screw. The amount by which it advances in one complete turn is called the *pitch* of the screw.

The thread is simply an inclined plane wound about the cylinder. If the surface of the cylinder be cut parallel to its axis and then rolled out flat, a single turn of the thread becomes a straight line inclined at an angle α to a circular section of the cylinder. If p is the pitch of the screw and r is its radius, it is seen from Fig. 84 that

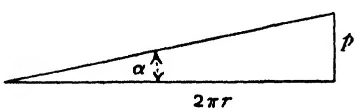


Fig. 84.

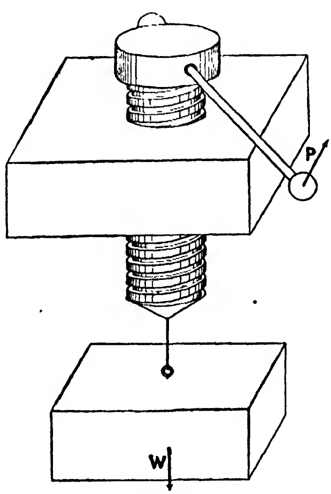


Fig. 85.

$$p = 2\pi r \tan \alpha.$$

Imagine a vertical screw which turns in a fixed nut. If the screw is attached to a weight W and is turned by means of an arm of length a, measured from the axis of the screw, what force applied at the end of the arm will hold the weight in equilibrium?

Let **P**, which is applied perpendicularly to the plane which passes through the axis of the screw and the end of the arm, balance the weight **W**. If the arm is given a small rotation $d\theta$, the amount of work done by **P** is $aPd\theta$. The height through which the weight is raised is $r \tan \alpha \cdot d\theta$. If there is no friction acting, the principle of virtual work gives the equation

$$aPd\theta - Wr \tan \alpha d\theta = 0$$

and therefore

$$P = \frac{r}{a} W \tan \alpha.$$

This analysis supposes that the weight of the screw itself is negligible, but if this is not so the weight of the screw must be included in W.

Consider now the effect of friction. Since the screw and the nut act uniformly at all points of the thread the frictional action can be regarded as concentrated at a single point. If **F** and **N** are the frictional and normal components of the action of the nut on the screw, the components parallel to the axis of the screw give the equation

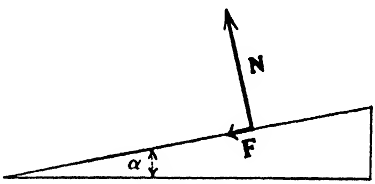


Fig. 86.

$$W = N \cos \alpha - F \sin \alpha$$

on the supposition that the screw is on the point of slipping upward. Therefore, since $F = N\mu$,

$$N = \frac{W}{\cos \alpha - \mu \sin \alpha} = \frac{W \cos \epsilon}{\cos (\alpha + \epsilon)}.$$

In the small rotation $d\theta$, the displacement between the nut and the screw is $r \sec \alpha d\theta$, and the work done against friction is $N\mu r \sec \alpha d\theta$. Hence, bearing in mind the value of N,

$$aPd\theta - Wr \tan \alpha d\theta - \frac{Wr\mu \sec \alpha d\theta}{\cos \alpha - \mu \sin \alpha} = 0;$$

from which is derived

$$P = \frac{r}{a} W \tan (\alpha + \epsilon).$$

If the screw is about to slip downward, it is necessary merely to change the sign of ϵ in the expression. Equilibrium evidently will exist for any value of P between the two extreme values so obtained; that is,

$$\frac{r}{a} W \tan (\alpha - \epsilon) \leqq P \leqq \frac{r}{a} W \tan (\alpha + \epsilon).$$

If the threads are V-shaped, with the angle at the V equal to 2β, then

$$\frac{r}{a}W\frac{\sin\alpha\cos\beta\cos\epsilon - \cos\alpha\sin\epsilon}{\cos\alpha\cos\beta\cos\epsilon + \sin\alpha\sin\epsilon} \leqq P$$
$$\leqq \frac{r}{a}W\frac{\sin\alpha\cos\beta\cos\epsilon + \cos\alpha\sin\epsilon}{\cos\alpha\cos\beta\cos\epsilon - \sin\alpha\sin\epsilon},$$

the proof of which will be left as an exercise.

Problems XII

(Use the principle of virtual work)

1. A system of pulleys is arranged in such a way that when the end of the rope is pulled 10 ft. a weight w is lifted 1 ft. What force is required to lift the weight?

2. Four rods of equal length l and weight w are freely jointed so as to form a rhombus $ABCD$. The corners A and C are connected by a string of length s, and the rhombus is suspended freely from the corner A. What is the tension in the string? *Ans.*

$$T = 2w.$$

3. A rhombus, similar to that of problem 2, but of negligible weight, is supported by two pegs in a horizontal line at a distance $2a$ apart touching AB and AD. A weight W is attached to the lowest corner C. The corners B and D are in a horizontal line and are connected by a string. If the angle at A is 2α, show that the tension in the string is

$$W\tan\alpha\left(\frac{a}{2l}\operatorname{cosec}^3\alpha - 1\right).$$

4. A heavy elastic string of natural length a, modulus λ, and weight w is placed upon a smooth vertical cone, the angle of the vertex of which is 2α. What is the distance of its plane of equilibrium from the vertex?

Ans. $$d = \frac{a}{2\pi\tan\alpha}\left(1 + \frac{w}{2\pi\lambda\tan\alpha}\right).$$

5. If a 10-lb. weight descending 4 ft. will drive a clock for 36 hr., what couple must be applied to the minute hand to hold it still if the pendulum and the escapement are removed? *Ans.* $M = 5/9\pi$ ft.-lb.

6. A weight w is supported by two strings, one of which is of length a and inextensible, while the other is elastic, modulus λ, and natural length a. The strings are tied to two pegs in a horizontal line distance a apart. Find the configuration of equilibrium. *Ans.* The angle α opposite the elastic string is defined by the equation

$$\left(2\sin\frac{1}{2}\alpha - 1\right)\cos\frac{1}{2}\alpha - \frac{w}{\lambda}\cos\alpha = 0.$$

7. A screw with a pitch p makes an angle α with the horizontal. If there is no friction, what couple is required to hold a nut of weight w from running down the screw? *Ans.* $\dfrac{wp\sin\alpha}{2\pi}$.

8. If every particle of a uniform hoop of radius r is repelled from the center by a force ma^2, where m is the mass of the particle and a is a constant, show that the tension in the hoop is $\sigma r a^2$, where σ is the mass of the hoop per foot.

9. A square board of sides a and weight w is supported by four light rods of length l which are vertical and which are freely hinged at points in the ceiling and at the corners of the board. The board is rotated through an angle θ and a string is tied around the four middle points of the rods (which also form a square). What is the tension in the string, if the system is in equilibrium? *Ans.*

$$T = \frac{wa\cos\frac{1}{2}\theta}{4\sqrt{l^2 - a^2\sin^2\frac{1}{2}\theta}}.$$

10. The legs of a tripod are of length l and make an angle with the vertical equal to θ. However much the legs of the tripod are spread, they always form an equilateral triangle. The tripod rests on a smooth plane and its legs are kept from spreading by an endless string around the feet. If the total weight of the tripod and the load which it supports is w, show that the tension in the string is

$$T = \frac{w}{3\sqrt{3}}\tan\theta.$$

11. Many of the problems in previous sets are solved more easily by the principle of virtual work. The student is recommended to try some of these.

V. STABLE AND UNSTABLE EQUILIBRIUM

168. Different Types of Equilibrium.—In accordance with the principles of equilibrium which have been set forth, a lead pencil standing upon a sharpened point with its center of gravity in the same vertical line with its point is in equilibrium. Yet everyone knows that a pencil will not remain in such a position unless it is otherwise supported.

On the other hand, a pencil can be suspended from its point by a fine thread. If the center of gravity and the point of suspension are in the same vertical line, the pencil is again in equilibrium, but this time the position of the pencil will be maintained. This equilibrium is said to be *stable*. In the first case, the equilibrium is said to be *unstable*.

Finally, if the pencil is laid on its side on a flat, horizontal table, the pencil is in equilibrium in every position. In this case, the equilibrium is said to be *neutral*.

In the first case, the center of gravity is at the greatest possible height, for, if the point of the pencil be kept fixed, whatever position the pencil may have the center of gravity lies on a

sphere of which the point is the center. The maximum height occurs when the pencil is vertical. This is also the position for which the potential energy (Sec. 130) is a maximum. In the second case, if the string is taut and in the same straight line with the pencil, the center of gravity of the pencil lies on the surface of a sphere which has the point of suspension of the thread as its center. If the pencil and thread are not in a straight line, the center of gravity of the pencil lies inside of this bounding sphere. In the position of equilibrium, that is, when the pencil hangs straight down, the center of gravity is at its lowest possible point and the potential energy is a minimum. In the third

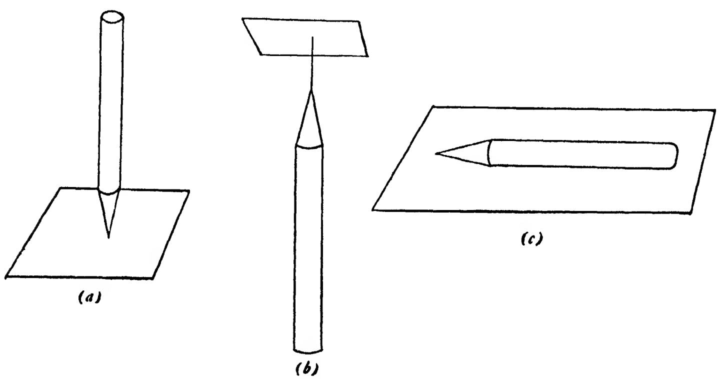


Fig. 87.

case, the height of the center of gravity is constant and, therefore, the potential energy also is constant. It will be shown that these conditions are characteristic of the three states of equilibrium.

169. Relation between Potential Energy and Equilibrium.— Suppose a body, or a system of bodies, is given which is acted upon by conservative forces and which has n degrees of freedom. Its potential energy is then a function of n parameters:

$$\text{potential energy} = V(q_1, q_2, \cdots, q_n).$$

The parameters may be rectangular coordinates or polar coordinates, or they may have other interpretations. Let one of the parameters, say q_1, be given a small increment dq_1, the others being kept fixed. The increase in the potential energy is

$$dV = \frac{\partial V}{\partial q_1} dq_1$$

and, from the definition of potential energy, this is the amount of work which is done upon the body in making the displacement dq_1. Since the work done is the product of the force and the displacement, the force which acts during the displacement is $\partial V/\partial q_1$, and the force which acts naturally upon the body is $-\partial V/\partial q_1$ in the q_1-direction, if permission is granted to use the word *direction* in this generalized sense. But in a position of equilibrium, the resultant of the forces acting is zero in every direction and, therefore,

$$\frac{\partial V}{\partial q_1} = 0, \quad \frac{\partial V}{\partial q_2} = 0, \quad \cdots, \quad \frac{\partial V}{\partial q_n} = 0;$$

that is, all of the first derivatives of the potential energy vanish at a position of equilibrium of a body which is acted upon by conservative forces.

Conversely, if all of the first derivatives vanish for a given set of values of $q_1, q_2, \cdots q_n$, then this set of values of the parameters defines a position of equilibrium of the body, for the sum of the components of the forces acting vanishes in every direction.

170. Illustrative Example.—Two uniform rods each of length $2l$ and weight w connected at one of their extremities by a smooth hinge are thrown across a smooth horizontal cylinder of radius a, but are constrained to remain in a vertical plane. It is required to find the configuration of equilibrium.

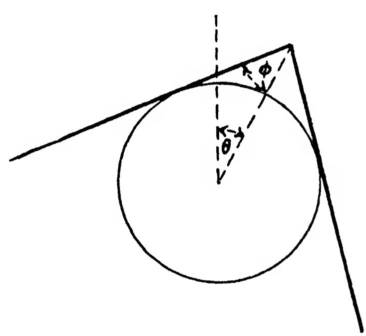


FIG. 88.

Let the line (Fig. 88) which joins the center of the cylinder to the hinge make an angle φ with each of the rods and an angle θ with the vertical. The center of gravity of the two rods lies on this line and on the line which joins the two midpoints of the rods. Its height above the axis of the cylinder is

$$h = (a \operatorname{cosec} \varphi - l \cos \varphi) \cos \theta.$$

The potential energy of the system of rods, therefore, as long as they are in contact with the cylinder, is

$$V(\varphi, \theta) = 2w(a \operatorname{cosec} \varphi - l \cos \varphi) \cos \theta.$$

The hinge of the rods must be above the cylinder and not on one side of it; otherwise, the constraint of the cylinder will cease to act. This condition implies that

$$\theta < \varphi < \frac{\pi}{2}.$$

The conditions for equilibrium are

$$\frac{\partial V}{\partial \varphi} = 2w\left(l \sin \varphi - a \frac{\cos \varphi}{\sin^2 \varphi}\right) \cos \theta = 0,$$

and

$$\frac{\partial V}{\partial \theta} = 2w(l \cos \varphi - a \operatorname{cosec} \varphi) \sin \theta = 0.$$

Since $-\pi/2 < \theta < \pi/2$, these conditions can be satisfied only by

$$\theta = 0, \qquad l \sin \varphi - a \frac{\cos \varphi}{\sin^2 \varphi} = 0;$$

or

$$\theta = 0, \qquad f(\varphi) = l \sin^3 \varphi - a \cos \varphi = 0.$$

Since

$$f(0) = -a, \qquad f\left(\frac{\pi}{2}\right) = +l,$$

$$\frac{df}{d\varphi} = 3l \sin^2 \varphi \cos \varphi + a \sin \varphi > 0,$$

there is one and only one value of $\varphi = \varphi_0$ in the first quadrant which satisfies the conditions and, in the configuration of equilibrium, the hinge of the rods is above the axis of the cylinder and the angle between the rods is $2\varphi_0$.

171. Maxima and Minima of the Potential Energy.—For the sake of simplicity, let the body have but one degree of freedom. Let q be the parameter which expresses this freedom which may be a translation, a rotation, a rolling, etc. Thus, as has been stated, the potential energy is a function of this parameter which can be expanded by Taylor's theorem, *viz.*,

$$V(q) = V(q_0) + \left.\frac{\partial V}{\partial q}\right|_{q=q_0} (q - q_0) + \left.\frac{1}{2!}\frac{\partial^2 V}{\partial q^2}\right|_{q=q_0} (q - q_0)^2 + \cdots,$$

where q_0 is any ordinary value of the parameter q. Let $q = q_0$ be a value of the parameter which corresponds to a position of equilibrium. Then the first derivative vanishes, *i.e.*,

$$\left.\frac{\partial V}{\partial q}\right|_{q=q_0} = 0.$$

But this is the condition that $V(q)$, considered merely as a function of q, shall have a maximum or a minimum at $q = q_0$. It follows, therefore, that *a position of equilibrium is a position in which the potential energy, if it is not a constant, is either a maximum, a minimum, or a minimax.* If the second derivative

$$\left.\frac{\partial^2 V}{\partial q^2}\right|_{q=q_0}$$

is positive, $V(q_0)$ is a minimum; if it is negative, $V(q_0)$ is a maximum; and if it is zero, but the third derivative not zero, $V(q_0)$ is a minimax.

172. Direction of Force Near Position of Equilibrium.—It has been observed that the force acting upon the system is given by $-\partial V/\partial q$, which, expanded by Taylor's theorem, is

$$-\frac{\partial V}{\partial q} = -\left.\frac{\partial V}{\partial q}\right|_{q=q_0} - \left.\frac{\partial^2 V}{\partial q^2}\right|_{q=q_0}(q - q_0) - \cdots.$$

If $q = q_0$ is a position of equilibrium, the first term of the right member vanishes and the expression becomes

$$\text{force} = -\frac{\partial V}{\partial q} = -\left.\frac{\partial^2 V}{\partial q^2}\right|_{q=q_0}(q - q_0) - \cdots.$$

If

$$\left.\frac{\partial^2 V}{\partial q^2}\right|_{q=q_0} > 0,$$

this expression shows that the force is opposite in sign to the displacement $q - q_0$ and, therefore, tends to make the body return to the position of equilibrium if it is slightly displaced, and the equilibrium is stable. If, however,

$$\left.\frac{\partial^2 V}{\partial q^2}\right|_{q=q_0} < 0,$$

the force has the same sign as the displacement $q - q_0$ and, therefore, tends to exaggerate the displacement. Accordingly, the body moves away from the position of equilibrium and the equilibrium is unstable.

A similar argument shows that if $q = q_0$ is a minimax of $V(q)$, the force is directed toward the position of equilibrium on one side of it and away from the position of equilibrium on the other side. Hence, on one of the two sides, the body tends to move

away from the position of equilibrium. Since it is necessary for stability that the body should tend to return to the position of equilibrium for *every* small displacement which the body may have, a minimax in the potential energy denotes a position of unstable equilibrium.

Theorem.—If the potential energy of a position of equilibrium is either a maximum or a minimax, the equilibrium is unstable. If the potential energy is a minimum the equilibrium is stable.

Since the maxima and minima of a continuous function occur alternately, so also do the positions of stable and unstable equilibrium of a body with one degree of freedom occur alternately.

If the potential energy V is a constant, then $\partial V/\partial q$ is zero everywhere and the equilibrium is neutral. No force acts in the direction in which the body is free to move.

It is evident from the above discussion that if a body is at rest in a conservative field of force but not in equilibrium, it starts to move in such a way that its potential energy decreases.

173. Systems with Many Degrees of Freedom.—If a system of particles has many degrees of freedom, the potential function will contain as many variables, in general, as there are degrees of freedom. The argument for stability or instability is similar in character to that for a single degree of freedom, but the mathematics are more complicated. A position of equilibrium may be stable for certain variables (or displacements) and unstable for others. Such positions are unstable positions of equilibrium. In order that a position of equilibrium may be stable, it must be stable for every possible small displacement compatible with the constraints. This means that the potential function must be a minimum with respect to each of the variables.

174. Illustrative Example.—Examine the character of the equilibrium of the problem in Sec. 170. In this problem, it was found that

$$V(\varphi, \theta) = 2w(a \operatorname{cosec} \varphi - l \cos \varphi) \cos \theta$$

and that the position of equilibrium is defined by

$$\theta = 0, \qquad \varphi = \varphi_0 < \frac{\pi}{2},$$

where $$f(\varphi_0) = l \sin^3 \varphi_0 - a \cos \varphi_0 = 0.$$

With respect to changes in the angle φ, it is found that

$$\frac{\partial^2 V}{\partial \varphi^2} = 2w\left(l \cos \varphi + a\frac{2 \cos^2 \varphi + \sin^2 \varphi}{\sin^3 \varphi}\right) \cos \theta,$$

which is positive for $\theta = 0$ and $\varphi = \varphi_0$, and therefore V is a minimum with respect to φ alone. Also,

$$\frac{\partial^2 V}{\partial \theta^2} = 2w(l \cos \varphi - a \operatorname{cosec} \varphi) \cos \theta$$

which is positive or negative for $\theta = 0$ and $\varphi = \varphi_0$, according as

$$l \cos \varphi_0 - a \operatorname{cosec} \varphi_0 \gtrless 0.$$

But this expression is the negative of the height of the center of gravity of the two rods above or below the center of the cylinder in the position of equilibrium. If the center of gravity is above the center of the cylinder,

$$l \cos \varphi_0 - a \operatorname{cosec} \varphi_0 < 0;$$

the potential energy has a maximum and the equilibrium is unstable. If the center of gravity is below the center of the cylinder,

$$l \cos \varphi_0 - a \operatorname{cosec} \varphi_0 > 0;$$

the potential energy has a minimum, and the equilibrium is stable both for changes in φ and θ. Hence, the equilibrium is stable only if

$$l \cos \varphi_0 - a \operatorname{cosec} \varphi_0 > 0.$$

Problems XIII

1. A rod is supported in a horizontal position by a light string attached to its two ends and passing over a smooth peg. Is its position stable? *Ans.* No.

2. A light string is attached to a fixed point O and passes over a fixed pulley P, so that $OP = 2a$ is horizontal, and supports a weight W_1 attached to its end. A second weight $W_2 < 2W_1$ slides freely on the string between O and P. Find the position of equilibrium and show that it is stable.

Ans. Distance of W_2 below OP is $\dfrac{aW_2}{\sqrt{4W_1^2 - W_2^2}}$.

3. The axis of a smooth parabolic wire is vertical. Two similar beads are strung on it and are connected by a light string which passes over a smooth peg at the focus. Show that the equilibrium of the beads is neutral.

4. The center of gravity of an elliptic disk is halfway between the center of the disk and one end of the major axis. Show that if the eccentricity is greater than $1/\sqrt{2}$, there are four positions of equilibrium and discuss their stabilities.

5. Prove that a tumbler of thin glass composed of a cylindrical part and a hemispherical base of the same radius will stand upright on a table if the height of the tumbler does not exceed its diameter; and that a liquid poured into it will not disturb the equilibrium before it reaches the cylindrical part.

6. A solid hemisphere with its plane face uppermost rests on a rough solid sphere of the same radius. Is its equilibrium stable or unstable if it cannot slide? *Ans.* Unstable.

7. A rod is thrown into a smooth elliptic bowl of semiaxes $a > b > c$. In what position does it come to rest?

8. A solid cylinder, resting on its base, is tipped sideways through various angles. Construct the curve of its potential energy and determine its positions of equilibrium graphically.

VI. FRAMEWORKS

175. Rigid Frameworks.—A series of rods connected only by hinges at their extremities forms a framework which may or may not be rigid. Three rods in the shape of a triangle form a rigid framework, while four rods in the shape of a quadrilateral do not. If a fifth rod be inserted as a diagonal of the quadrilateral, the framework becomes just rigid. If a sixth rod be inserted as the other diagonal, the framework becomes overrigid. A framework is *just rigid* if no rod of the framework can be removed without destroying its rigidity and is *overrigid* if any rod whatever can be removed without destroying its rigidity. Any such superfluous rod is called a *redundant rod.* If the framework is a quadrilateral with the two diagonals, any one of the six rods is redundant.

176. Light Frameworks.—If the weight of the rods themselves can be neglected, and if the hinges are *smooth,* the framework is said to be a *light framework.* Suppose that such a light framework is given and that it is acted upon by given forces at the hinges. It is desired to find the magnitudes of the resulting tensions or thrusts of the rods of the framework.

Let $ABCD$ (Fig. 89) be the framework acted upon by the known forces $\mathbf{F}_1$ and $\mathbf{F}_2$ at A and B, respectively; by the force $\mathbf{F}_3$, of which the direction only is known, at the point C; and by the wholly unknown force $\mathbf{F}_4$ acting at D. It is desired to determine $\mathbf{F}_4$, the magnitude of $\mathbf{F}_3$, and the tensions (or thrusts) in the rods.

177. The Stress Diagram.—The solution of this problem is very simply attained graphically by means of a system of polygons known as *the stress diagram* (Fig. 90).

At the hinge A, there are three forces acting of which $\mathbf{F}_1$ is wholly known. The lines of action of the other two are, of course, along the rods and therefore their directions are known. Since

A is in equilibrium, these three forces can be arranged in the form of a closed triangle which can actually be drawn since $\mathbf{F}_1$ is known. Let abc (Fig. 90) be this triangle, drawn to a suitable scale. If $\overline{ab}$ represents $\mathbf{F}_1$, then $\overline{bc}$ and $\overline{ca}$ will represent the actions of the rods 1 and 2 on the hinge A, if the lines bc and ca are drawn parallel to the rods 1 and 2, respectively. The reactions of the hinge A on the rods are thrusts represented by $\overline{cb}$ and $\overline{ac}$, which are now known. It is convenient to number the sides cb and ac, 1 and 2,

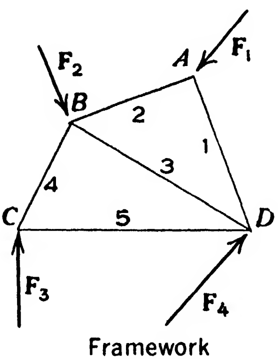


FIG. 89.

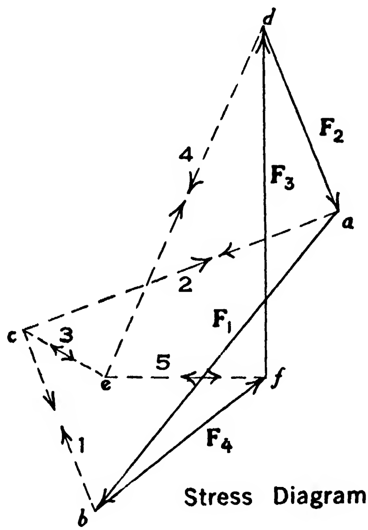


FIG. 90.

respectively, since they represent the thrusts in the rods which are numbered 1 and 2.

Of the four forces which are acting upon the hinge B, two are wholly known and the lines of action of the other two are known. These four forces form a closed quadrilateral which now can be drawn. Let $\overline{da} = \mathbf{F}_2$, and let de and ce be drawn parallel to the rods 4 and 3, and numbered accordingly. Then $daced$ is a closed quadrilateral the sides of which, taken in succession, represent the forces which are acting upon the hinge B. From the direction of these forces $\overline{ce}$ represents a pull of the rod 3 on B and, therefore, $\overline{ec}$ represents a pull of the hinge B on the rod 3. The rod 3 is therefore under tension, while the rod 4 sustains a thrust which is equal to $\overline{de}$.

At the hinge C, one force now is known and the lines of action of the other two are known. Let ef be drawn parallel to rod 5, and fd parallel to $\mathbf{F}_3$. Then

$$\overline{fd} = \mathbf{F}_3$$

and $\overline{ef}$ represents the action of the rod 5 on C. Accordingly, rod 5 is under tension of which the magnitude is ef. Finally, since the entire system is in equilibrium,

$$\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 + \mathbf{F}_4 = 0$$

and, if placed end to end, these vectors form a closed polygon. Hence, the force $\mathbf{F}_4$ acting at D is represented by the line $\overline{bf}$.

178. Difficulties in Starting.—It will be observed that a start was made at a point where only three forces were acting,

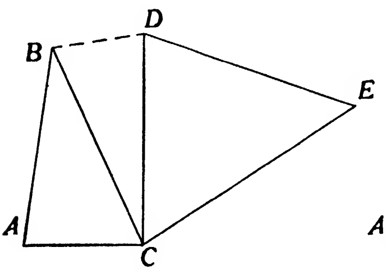


FIG. 91.

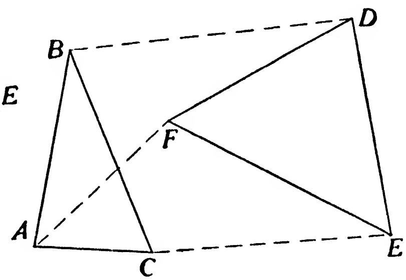


FIG. 92.

one of which was known. If no such point can be found some preliminary work may be necessary before a start can be made on the stress diagram (the term *stress* includes both thrusts and tensions). It may be that there is no single hinge at which a known force is acting, in which case it may be necessary to determine first all of the external forces which are acting on the framework before attempting to determine the internal stresses; or, if no single hinge occurs in the framework, it may be necessary to determine the stress in some one rod before a start can be made.

Two rigid triangles, for example, can be rigidly connected by means of a triple hinge and an extra rod, as in Fig. 91 or by means of three extra rods, as in Fig. 92. In this last framework (Fig. 92), there are no single hinges anywhere. The triangle DEF, however, can be regarded as by itself. If the moments of all of the forces acting upon it be taken about the intersection of AF and CE, the stress in BD will be determined.

179. Forces Not at the Hinges.—If forces act at points of the framework other than the hinges, they can be replaced by equivalent pairs of forces acting at the ends of the rods. This substitution will not alter the actions upon the hinges, but it will alter the stresses in the rods for which a change of forces is made. Thus, if the force **F** is acting at a point P of a rod (Fig. 93), let f_1 and f_2 be two cross-sections of the rod which contain P between them. The small portion of the rod between f_1 and f_2 is in equilibrium under the action of three forces, one of which is **F**, and the other two $\mathbf{F}_1$ and $\mathbf{F}_2$ are equivalent to the stresses across the planes f_1 and f_2. Evidently, $\mathbf{F}_1$ and $\mathbf{F}_2$ depend upon **F**, and therefore the internal stresses in the rod are altered if **F** is replaced by two equivalent forces at the ends of the rod.

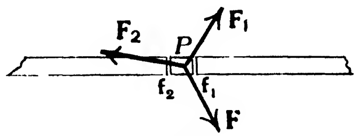


FIG. 93.

180. Heavy Frameworks.—If the rods which compose the framework are heavy, so that their weight cannot be neglected, the action upon the hinges can be determined by supposing that

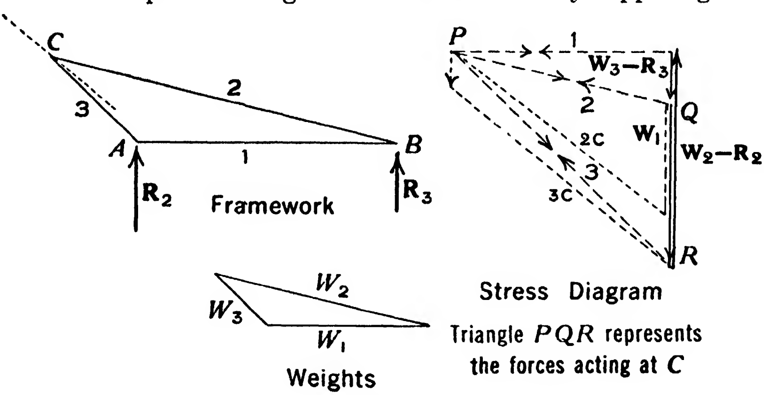


FIG. 94.

the rods are light and that vertical forces each equal to half the weight of the rods (if the rods are uniform) are acting at their extremities. The stress diagram will then give the forces acting upon the hinges. The stresses in the actual rods will vary from point to point on account of their weight (see Sec. 223).

Let ABC (Fig. 94) be a rigid triangle built of heavy beams smoothly jointed at the hinges. The lengths of the beams are l_1, l_2, and l_3, and their weights W_1, W_2, and W_3 are proportional

to their lengths. Let the weight of each beam be represented by two forces each equal to half of the weight of the beam acting at the hinges. Then it follows that,

$$\text{acting at } C, \qquad \mathbf{W}_1 = \frac{1}{2}(W_2 + W_3) = \frac{c}{2}(l_1 + l_2),$$

$$\text{acting at } A, \qquad \mathbf{W}_2 = \frac{1}{2}(W_3 + W_1) = \frac{c}{2}(l_2 + l_3),$$

$$\text{and} \quad \text{acting at } B, \qquad \mathbf{W}_3 = \frac{1}{2}(W_1 + W_2) = \frac{c}{2}(l_3 + l_1).$$

If the framework rests on supports at A and B, it is also acted upon by vertical forces of magnitude R_2 and R_3 at these points. Hence, $\mathbf{W}_1$ acts vertically at C, $\mathbf{W}_2 - \mathbf{R}_2$ at A, and $\mathbf{W}_3 - \mathbf{R}_3$ at B. The action of the light rods on the hinges is given by the lines 1, 2, and 3 of the stress diagram. Since W_1, W_2, and W_3 are known, R_2 and R_3 are determined. By compounding the stress 2 with $W_2/2$ acting vertically at C, the action of the beam 2 on the hinge is obtained. The resultant is the dotted line $2c$ in the stress diagram. By compounding the stress 3 with the weight $W_3/2$ acting vertically at C, the action of beam 3 on the hinge C is obtained. The resultant is marked $3c$ in the diagram. It will be observed that $2c$ and $3c$ are equal, parallel, and opposite as they should be, since no other force acts on the hinge at C.

The action of the beams on the other hinges can be obtained in a similar manner.

Problems XIV

1. A light framework in the shape of a triangle ABC with sides $AB = l_1$, $AC = l_2$, and $BC = l_3$ is fastened to a vertical wall at the corner C, touching the wall again only at the point B where the contact is smooth. A weight W is suspended from the point A. Find the forces acting at B and C and the stresses in the rods.

Ans. A tension in l_2 equal to Wl_2/l_3; a thrust in l_1 equal to Wl_1/l_3. If the distance of A from the wall is p, and $\tan\alpha = p/l_3$, the force at C makes an angle α with the wall, and its magnitude is $W \sec\alpha$. The force at B is normal to the wall and its magnitude is $W\tan\alpha$.

2. A light framework forms a right-angled isosceles triangle of sides $l_1 = l_2$ and l_3 and is suspended freely at the right angle. A weight W is suspended at the extremity of l_1 and a weight $2W$ at the extremity of l_2. Find the stresses in the rods.

Ans. Tensions in l_1 and l_2 equal to $3W/\sqrt{5}$ and $6W/\sqrt{5}$, respectively, and a thrust in l_3 equal to $2\sqrt{2}W/\sqrt{5}$.

3. A light truss built of 11 rods of the same length and smoothly jointed is supported at its extremities P_1 and P_4, and supports weights of 7 and 4 tons at the points P_2 and P_3. Find the stresses in the rods.

Ans. Thrusts in 1, 5, 7, 9, and 11 equal to 12, 12, 2, 10, and 10 tons, respectively; tensions in 2, 3, 4, 6, 8, and 10 equal to 6, 12, 2, 11, 10, and 5 tons, respectively; all divided by $\sqrt{3}$.

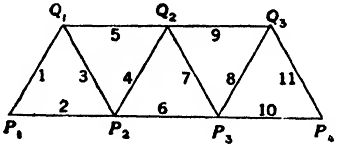


Fig. 95.

4. Five uniform beams form the sides of a rhombus $ABCD$ and the diagonal BD which is only half the lengths of the sides. If the weight of the diagonal beam is W and the frame is suspended by the hinge at A, find the action on the hinge at B of each beam joined to B.

Ans. BD exerts a thrust $1.16W$ and a downward force $W/2$;
AB exerts a tension $3.62W$ and a downward force W;
BC exerts a tension $1.03W$ and a downward force W.

General Problems XV

1. Four forces $\mathbf{F}_1, \cdots, \mathbf{F}_4$ act along the sides of a square. What is the condition that the line of action of the equivalent single force passes through the center of the square?

2. Four forces act along the sides of a quadrilateral and in magnitude are proportional to the sides along which they act. If equilibrium exists, show that the quadrilateral is a parallelogram.

3. Four forces act upon a rigid body and their lines of action form a quadrilateral. Show that the quadrilateral must be plane if the body is in equilibrium.

4. Forces **A**, **B**, and **C** act along the sides of a triangle a, b, and c and the line of action of the equivalent single force passes through the centers of the inscribed and circumscribed circles. Show that

$$\frac{A}{\cos b - \cos c} = \frac{B}{\cos c - \cos a} = \frac{C}{\cos a - \cos b}.$$

5. Four forces act along the sides of a quadrilateral and in magnitude are equal to a, b, c, and d times those sides. If equilibrium exists, show that $ac = bd$ and that the diagonals are divided by their points of intersection in the ratios

$$a:b \quad \text{and} \quad c:d.$$

6. If the quadrilateral, in the previous problem, can be inscribed in a circle and if the magnitudes of the forces are inversely proportional to the lengths of the opposite sides, show that the line of action of the equivalent single force passes through the intersections of the pairs of opposite sides.

7. If, in the previous problem, equilibrium exists, show that the magnitudes of the forces are directly proportional to the opposite sides.

8. The ratio of the moments of a force of given magnitude, but of variable direction and point of application, about two fixed points A and B, is con-

stant. Show that the line of action of the force always passes through a fixed point which lies on the line joining A and B.

9. The ratios of the moments of a force about three points, which are not collinear, are given. Find a graphical construction for the line of action of the force.

10. The moments of a force with respect to the points (1, 0), (0, 2), and (2, 3) are c_1, c_2, and c_3. What angle does the force make with the x-axis?

Ans.
$$\tan\theta = \frac{c_1 - 3c_2 + 2c_3}{2c_1 - c_2 - c_3}.$$

11. A rod of length $2l$ lies on a smooth table and has attached to it at its ends and at its middle point light strings to which are tied weights W_1, W_2, and $W_1 + W_2$. The first two strings hang over one edge of the table and the third over the opposite parallel edge. If the rod is kept in equilibrium at an angle θ to these edges by two smooth pegs fixed to the table at a distance a apart, find the pressures on the pegs. *Ans.* Each pressure is $l(2W_1 + W_2)\cos\theta/a$.

12. A smooth hemispherical bowl of radius r is held firmly. A uniform rod rests against the rim with one end on the inner surface of the bowl. If the length of the rod within the bowl is c, show that the length of the entire rod is $4(c^2 - 2r^2)/c$.

13. A non-uniform rod of weight w_1 and of length $2a$ is pivoted at its middle point which is at a distance h from its center of gravity. A weight w_2 slides on a string of length $2l$ the ends of which are fastened to the ends of the rod. Find the distances of w_2 from the ends of the rod in the position of equilibrium. *Ans.*

$$\frac{aw_2 - hw_1}{aw_2}\cdot l \quad \text{and} \quad \frac{aw_2 + hw_1}{aw_2}\cdot l.$$

14. Equal forces act on a circular disk, one at a point A of the edge and the other at the center O in a direction perpendicular to OA. Prove that they cannot be balanced by any force whose point of application lies within the disk, unless the angle θ between the direction of the equal forces satisfies the relation.

$$\cos^2\theta < 4\cos^2\frac{1}{2}\theta.$$

15. A square board is suspended from one corner by a string, equal in length to the side of the square, which is attached to a point of a smooth vertical wall. If the plane of the board is perpendicular to the wall, show that in the position of equilibrium the distances of the corners of the square from the wall are proportional to 0, 1, 4, and 3.

16. If a smooth sphere whose center of gravity is not at its center rests in contact with any number of smooth surfaces, show that the line joining its center to its center of gravity must be vertical in the position of equilibrium.

17. Three equal smooth spheres are held in contact with one another on a horizontal plane by a band around them. If a fourth sphere, similar to

the other three, is placed upon them, show that the tension of the band is increased by $W/\sqrt{54}$.

18. A pentagon $ABCDE$, formed of equal, uniform, heavy rods connected by smooth hinges at their ends, is supported symmetrically in a vertical plane with A uppermost and AB and AE in contact with two smooth pegs in the same horizontal line. Prove that if the pentagon is regular, the pegs divide AB and AE each in the ratio

$$1 + \sin\frac{\pi}{10} : 3\sin\frac{\pi}{10}.$$

19. Two equal and similar isosceles wedges, each of weight W_1 and vertical angle 2α, are placed side by side with their bases on a rough horizontal plane with their edges in contact. A smooth sphere of weight w_2 and radius r in contact with a face of each is supported between them. If equilibrium exists, prove that

$$\mu > \frac{w_2 \cot\alpha}{2w_1 + w_2} \quad \text{and} \quad r < a \sin\alpha \tan\alpha \left(\frac{w_1 + w_2}{w_2}\right),$$

where μ is the coefficient of friction and a is the length of each base.

20. Two equal, smooth, circular disks of radius r are placed on their flat sides in a corner between two smooth vertical planes, inclined at an angle 2α, and touch each other in the line bisecting the angle. Show that the smallest disk which can be pressed between them without causing them to separate is one of radius

$$\rho = r(\sec\alpha - 1).$$

21. One end of a straight uniform beam rests on a rough floor, the other end being connected by a string to a point in the ceiling in the vertical plane which contains the beam. If θ_1, θ_2, and θ_3 are the inclinations of the string, the beam and the reaction of the floor show that

$$\cot\theta_1 \pm 2\cot\theta_2 - \cot\theta_3 = 0.$$

22. A right circular cone of vertical angle 2α rests with its base on a rough horizontal plane. A string is attached to its vertex and pulled horizontally with a gradually increasing force. In what way will equilibrium first be broken? *Ans.* It will slide if $\epsilon < \alpha$ and will tip if $\epsilon > \alpha$.

23. A heavy bar of length $4l$ and weight w is suspended by its extremities by two equal strings of length s which are attached to two points in the ceiling at a distance of $2l$ apart. If the bar, remaining horizontal, is turned through an angle θ, what torque is required to hold it in this position?

Ans. $$C = \frac{2ws\sin^2\varphi\sin\theta}{\sqrt{\cos^2\varphi - 8\sin^2\varphi\sin^2\frac{1}{2}\theta}}, \qquad \text{where } \cos\varphi = \frac{l}{s}.$$

24. A tight nut requires a force of 132 lb. applied at the end of a wrench 18 in. in length to move it. If the bolt on which the nut is screwed has a diameter of 1 in., if the nut is 3/4 in. in length, and if the coefficient of friction is 0.3, what is the pressure of the nut on the bolt? *Ans.* 280 lb. per square inch.

25. Prove that the central axis of two forces $\mathbf{F}_1$ and $\mathbf{F}_2$ intersects the shortest line, of length L, joining their lines of action and divides it in the ratio

$$F_1(F_1 + F_2 \cos \widehat{\mathbf{F}_1\mathbf{F}_2}):F_2(F_2 + \cos \widehat{\mathbf{F}_1\mathbf{F}_2}),$$

and that the moment of the equivalent wrench is

$$\frac{LF_1F_2 \sin \widehat{\mathbf{F}_1\mathbf{F}_2}}{\sqrt{F_1^2 + 2F_1F_2 \cos \widehat{\mathbf{F}_1\mathbf{F}_2} + F_2^2}}.$$

26. A heavy solid cylinder of radius r and weight w rests on a rough horizontal plane. Assuming the pressure on the base to be uniform and the cylinder and the plane rigid, show that the torque required to turn the cylinder is $2\mu wr/3$.

CHAPTER IX

STATICS OF DEFORMABLE BODIES

181. Introduction.—The bodies dealt with in the preceding chapter were rigid in the sense that they maintained their shape under all conditions. There are no such bodies in nature, of course, but the ideal serves as a very useful first approximation to many actual situations where the variations in the forces acting are small. A second approximation will be taken up in the present chapter in the consideration of bodies that are only slightly deformed. In addition to such cases, there are other bodies or systems of bodies that are freely deformable, such as systems of weights connected by light strings, ropes, chains, etc. The funicular polygon will be considered first.

I. FUNICULAR POLYGONS

182. The Funicular Polygon.—To a light string which is tied to two fixed points, a number of weights are attached. When the system is in equilibrium the segments of the string form the sides of a polygon which is known as the funicular polygon (from *funicle*, a light string).

In Fig. 96, let P_1 and P_2 be the points of suspension which are connected by a light string to which four weights W_1, W_2, W_3, and W_4 are attached. Let the lengths of the five segments of the string be l_1, l_2, l_3, l_4, and l_5; and let these segments make angles θ_1, θ_2, θ_3, θ_4, and θ_5 with the horizontal toward the right, so that the first of the angles is negative and the last is positive. Let the corresponding tensions in the strings be $\mathbf{T}_1$, $\mathbf{T}_2$, $\mathbf{T}_3$, $\mathbf{T}_4$, and $\mathbf{T}_5$. Finally, let the coordinates of the point P_2, with respect to the point P_1 as origin, be a and b.

On resolving horizontally and vertically the forces which act upon the point of the string to which the weight W_1 is attached, the following equations are obtained:

$$T_1 \cos \theta_1 = T_2 \cos \theta_2,$$
$$-T_1 \sin \theta_1 + T_2 \sin \theta_2 = W_1.$$

These equations show that the horizontal component of the tensions in the first two segments are equal. Similar equations for the weight W_2,

$$T_2 \cos \theta_2 = T_3 \cos \theta_3,$$
$$-T_2 \sin \theta_2 + T_3 \sin \theta_3 = W_2,$$

show that the horizontal component of the tension in the third segment is the same as for the first two. A continuation of the

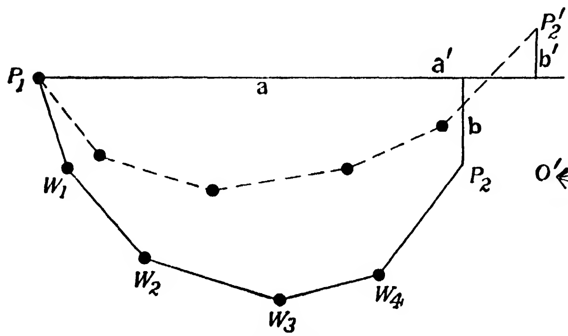


FIG. 96.

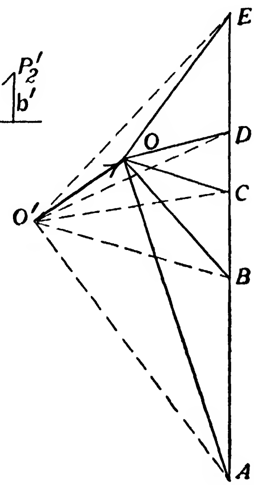


FIG. 97.

analysis shows that it is the same for all of the segments. Suppose the string has $n + 1$ segments and n weights. If l_i and l_{i+1} are two consecutive segments, it is evident that

$$T_i \cos \theta_i = T_{i+1} \cos \theta_{i+1} = H,$$
$$-T_i \sin \theta_i + T_{i+1} \sin \theta_{i+1} = W_i,$$

the common value of the horizontal components being denoted by the letter H. Hence,

$$T_i = \frac{H}{\cos \theta_i},$$
$$-\tan \theta_i + \tan \theta_{i+1} = \frac{W_i}{H}.$$

These equations hold however many weights there may be.

The sum of the horizontal projections of the segments themselves is the horizontal coordinate of P_2, and the sum of the vertical

projections of the segments is the vertical coordinate of P_2. Hence, the following system of equations is derived:

$$-\tan\theta_i + \tan\theta_{i+1} = \frac{W_i}{H}, \qquad i = 1, \cdots, n. \tag{1}$$

$$l_1\cos\theta_1 + l_2\cos\theta_2 + \cdots + l_{n+1}\cos\theta_{n+1} = a, \tag{2}$$

$$l_1\sin\theta_1 + l_2\sin\theta_2 + \cdots + l_{n+1}\sin\theta_{n+1} = b, \tag{3}$$

so that there are $n + 2$ equations with which to determine the $n + 2$ unknowns $\theta_1, \theta_2, \cdots, \theta_{n+1}$ and H.

183. The Force Diagram.—On a vertical line (Fig. 97), mark off the segments

$$AB = W_1, \quad BC = W_2, \quad CD = W_3, \quad DE = W_4.$$

Through the point A draw a line parallel to l_1, and through B draw a line parallel to l_2. These two lines meet in a point O, and the lines OA and OB are equal in magnitude to $\mathbf{T}_1$ and $\mathbf{T}_2$; for $\mathbf{W}_1$, $\mathbf{T}_1$, and $\mathbf{T}_2$, which are the forces acting on the point of attachment of W_1, if placed end to end, form a closed triangle. Since AB is equal and parallel to $\mathbf{W}_1$, and OA and OB are parallel to $\mathbf{T}_1$ and $\mathbf{T}_2$, respectively, they are equal in magnitude to T_1 and T_2. Similarly, OC, OD, and OE are equal to T_3, T_4, and T_5, respectively, and are parallel to $\mathbf{T}_3$, $\mathbf{T}_4$, and $\mathbf{T}_5$. Finally, since

$$H = T_i\cos\theta_i,$$

it is evident that H is the perpendicular distance from O to AE.

184. The Configuration of Equilibrium.—If the position of equilibrium of the system is not known, it is not known immediately how to locate the point O in the force diagram. But if any point O' is taken and the corresponding force diagram is drawn, it is a simple matter to draw the funicular polygon which is associated with it. From P_1 draw a line parallel to $O'A$ with a length l_1. Through the extremity of this line, draw a second line of length l_2 parallel to $O'B$, and so on, arriving finally at a point P_2'. If the system be suspended from the points P_1 and P_2', it will take the position which has just been drawn, and $O'ABCDE$ is its force diagram.

Let a point O' at a distance H' from AE be chosen, and let the corresponding angles θ_i' be computed so they can be regarded as known. The position of the point O could be determined if the components of the vector $\overrightarrow{O'O}$ were known. Let the magnitude of

the horizontal component be $x \cdot H'$, and the magnitude of the vertical component be $y \cdot H'$. Let O'' be a point directly above O' and at a distance $y \cdot H'$ from it, and let the force diagram be drawn for the point O''. The angles θ_i'' are related to the angles θ_i' in such a way that

$$H'' \tan \theta_i'' = H' \tan \theta_i' - H'y \qquad \text{and} \qquad H'' = H';$$

whence $\tan \theta_i'' = \tan \theta_i' - y \qquad i = 1, \cdots, n + 1.$

The points O and O'' are on the same perpendicular to AE. It is easily seen from the two force diagrams that

$$H \tan \theta_i = H'' \tan \theta_i'', \qquad H = (1 - x)H'';$$

whence,

$$\tan \theta_i = \frac{\tan \theta_i''}{1 - x}.$$

But these two transformations taken successively are equivalent to the single transformation

$$\tan \theta_i = \frac{\tan \theta_i' - y}{1 - x}, \qquad H = (1 - x)H';$$

from which it is found that

$$\sin \theta_i = \frac{\tan \theta_i' - y}{\sqrt{(1 - x)^2 + (\tan \theta_i' - y)^2}},$$

$$\cos \theta_i = \frac{1 - x}{\sqrt{(1 - x)^2 + (\tan \theta_i' - y)^2}}.$$

After substituting these values for $\sin \theta_i$ and $\cos \theta_i$ in Eqs. (182.2) and (182.3), there results

$$\sum_{i=1}^{n+1} \frac{l_i(1 - x)}{\sqrt{(1 - x)^2 + (\tan \theta_i' - y)^2}} = a,$$

$$\sum_{i=1}^{n+1} \frac{l_i(\tan \theta_i' - y)}{\sqrt{(1 - x)^2 + (\tan \theta_i' - y)^2}} = b.$$

This pair of simultaneous equations determines x and y, since the l_i, θ_i', a, and b are known. They are algebraic, but in general highly irrational. They can be solved when the numerical values are given, but the process is laborious.

185. Systems with Equal Weights.—If the weights are all equal, say equal to W, and if, for the sake of notation, the substitutions

$$\frac{W}{H} = c \qquad \text{and} \qquad \tan \theta_1 = t,$$

are made, then the equations of Sec. 182 become

$$\begin{aligned}
\tan\theta_1 &= t,\\
\tan\theta_2 &= t + c,\\
\tan\theta_3 &= t + 2c,\\
&\cdots\cdots\\
\tan\theta_n &= t + (n-1)c;
\end{aligned}$$

that is to say, the series of tangents forms an arithmetic series.

Let the first point of suspension P_1 be taken as the origin of a system of rectangular coordinates. Let x_i and y_i be the coordinates of the i^{th} weight, and finally, let

$$l_i = h \sec\theta_i,$$

so that the projections of the segments of the string upon the x-axis are all equal. Then

$$\begin{aligned}
x_1 &= h, & y_1 &= ht & &= ht,\\
x_2 &= 2h, & y_2 &= y_1 + h(t+c) & &= 2ht + hc,\\
x_3 &= 3h, & y_3 &= y_2 + h(t+2c) & &= 3ht + 3hc,\\
&\cdots & &\cdots & &\cdots\\
x_n &= nh, & y_n &= y_{n-1} + h(t+[n-1]c) & &= nht + \frac{n(n-1)}{2}hc.
\end{aligned}$$

The elimination of n between the expressions for x_n and y_n results in the equation

$$y_n = tx_n + \frac{1}{2}cx_n\left(\frac{x_n}{h} - 1\right).$$

This is the equation of a parabola with a vertical axis which passes through the origin and through all of the weights. By the substitutions

$$x_n = x - \left(t - \frac{1}{2}c\right)\frac{h}{c}, \qquad y_n = y - \left(t - \frac{1}{2}c\right)^2\frac{h}{2c},$$

the equation takes the normal form

$$y = \frac{c}{2h}x^2, \qquad \text{or} \qquad y = \frac{W}{2hH}x^2.$$

186. The Suspension Bridge.—These results have immediate application in the form taken by the cables of a suspension bridge. If the platform of the bridge is supported by wires equally spaced, there will be a uniform distribution of the weight and the horizontal projections of the segments of the cable between the wires will all be equal. If w is the weight carried by each of the verti-

cal wires, of which there are n for each cable, two in number, and h is the horizontal distance between the wires, then

$$nw = 2W \qquad \text{and} \qquad (n + 1)h = L,$$

where $4W$ is the total weight of the platform of the bridge and L is its length. Hence, the equation of the cable, referred to its lowest point as origin, is

$$y = \frac{W}{LH}x^2.$$

The quantity H is the horizontal component of the tension, and since this is constant, it is the tension of the cable itself at its lowest point. Let S be the sag of the cable, so that, in Fig. 98, the sag is

$$S = \overline{OC}.$$

Then, from the equation of the parabola,

$$S = \frac{WL}{4H}, \qquad \text{or} \qquad H = \frac{WL}{4S}.$$

Consider the equilibrium of one-half the cable from O to A. Since the load is uniformly distributed, it acts halfway from O

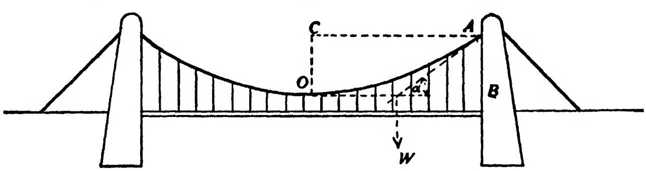


FIG. 98.

to B. The equilibrium of the cable would not be disturbed if it became rigid. This rigid body then is acted upon by the horizontal tension **H**, acting at O, the tension **T**, acting at A, and the weight **W**, acting along the line $x = L/4$. Since equilibrium exists, these three forces meet at a point. The slope of the curve is

$$\frac{dy}{dx} = \frac{2W}{LH}x;$$

therefore, if α is the angle which **T** makes with the horizontal,

$$\tan \alpha = \frac{W}{H}.$$

By applying Lami's theorem to the three forces at their common point of intersection, it is found that

$$T = H \sec \alpha = W \operatorname{cosec} \alpha = \sqrt{H^2 + W^2}.$$

187. Systems of Hinged Rods.—Given a system of heavy rods connected at their extremities by smooth hinges and suspended from two points at which the hinges also are smooth. Required: the configuration of equilibrium.

It is not necessary to suppose that the rods are uniform in any respect, but it will be assumed that the hinges are smooth. Let $\mathbf{W}_i$ be the weight of the i^{th} rod acting at its center of gravity. Let $\mathbf{L}_i$ and $\mathbf{R}_i$ be the equivalent system of parallel forces acting at the left and right hinges. Then the system of heavy rods can be replaced by a system of light rods (or light strings, since only tensions are involved) and a system of weights acting at the hinges. The configuration of equilibrium, therefore, is the funicular polygon in which the i^{th} weight is

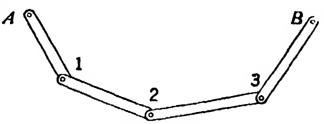


Fig. 99.

$$W_i = R_i + L_{i+1}.$$

The action on the hinge at A is obtained by adding vectorially the tension in the first rod to that portion of the weight of the first rod which acts at A; that is,

$$\mathbf{P}_A = \mathbf{T}_1 + \mathbf{L}_1;$$

and similarly, at B,

$$\mathbf{P}_B = \mathbf{T}_n + \mathbf{R}_n.$$

Problems XVI

1. The span of a suspension bridge is 200 ft., the sag of the cable is 20 ft. and its weight is 50 tons. What are the tensions at the highest and lowest points of the cable? *Ans.* 33.66 and 31.25 tons.

2. If the greatest tension permissible in the cable of the previous problem is 100 tons, what is the smallest permissible sag? (The greater the tension in the cable the steadier the bridge will be.) *Ans.* 6.3 ft.

II. CATENARIES

188. Flexible Chains and Ropes.—The same model which served as a light string in Chap. VII (Sec. 108) will also serve as a model of a chain or rope if the weight of the particles is taken into account. It is not necessary that the particles should all have the same weight. It is sufficient that the density of the chain varies from point to point continuously according to some specified law. The mean density of a section of the chain is the mass of that section divided by its length, or the mass per unit

length if the chain is uniform. The *density at a point* is the limit of the mean density of sections which contain that point, the sections being of diminishing length.

189. Conditions of Equilibrium under the Action of Gravity.— It will be supposed that a heavy chain is suspended from two fixed points by its extremities, and the form of the curve which it assumes when it is in equilibrium will be investigated.

Let ABC be such a chain hanging in equilibrium, and let B be its lowest point. If the portion of the chain AB were replaced by a horizontal force of magnitude H, equal to the tension which the part AB exerts on the point B, the remaining portion of the chain BC would be undisturbed. The external forces which are acting on the portion BC are then: a horizontal force **H** acting at B, a force **T** acting at C tangent to the curve, and the weight of its individual particles. Let s be the length of the arc as measured from B, and let $\sigma(s)$ be the density of the chain at the point whose distance along the curve from B is s.

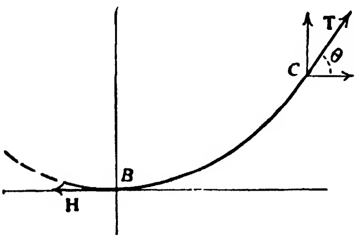


FIG. 100.

Let B be the origin of a system of rectangular axes of which the x-axis is horizontal and the y-axis is vertical. Then the equation of the curve assumed by the chain will be

$$y = f(x),$$

where $f(x)$ is some function which depends upon the law of density of the chain. The slope of the curve at C, which can be regarded as any point of the curve, is the value of dy/dx at the point C; that is,

$$\frac{dy}{dx} = \tan\theta.$$

Let it be imagined that the chain has become rigid in its position of equilibrium. This would not alter the shape of the curve nor the equilibrium, and it permits the application of the theorems of rigid bodies. On resolving the external forces which are acting upon the chain horizontally and vertically, it is found that

$$\left.\begin{aligned} H &= T\cos\theta, \\ W &= T\sin\theta, \end{aligned}\right\} \qquad (1)$$

where W is the weight of the section of the chain between B and C, and is therefore a function of s.

From the quotient of these equations, there is obtained:

$$\tan\theta = \frac{dy}{dx} = \frac{W}{H}. \tag{2}$$

If $\sigma(s)$ is the density of the chain at the distance s along the curve from B as defined in Sec. 188, and g is the acceleration of gravity, then

$$\sigma(s)\cdot g = w(s)$$

is the weight which a unit length of the chain would have if its density were constant and equal to $\sigma(s)$. It will be called the *weight of the chain per unit length at the point C.*

With this definition of $w(s)$, the weight of the chain from B to C is

$$W = \int_0^s w(s)ds. \tag{3}$$

Equations (2) and (3) hold wherever the point C may be along the arc of the curve. Hence, they determine the curve when the function $w(s)$ is given, or they determine $w(s)$ when the curve is given.

190. A Second Form of the Differential Equations.—On differentiating both equations of Eq. (189.1) with respect to s, and bearing in mind, from Eq. (189.3), that

$$dW = w\,ds,$$

it is found that

$$\sin\theta\cdot dT + T\cos\theta\cdot d\theta = w\,ds, \tag{1}$$
$$\cos\theta\cdot dT - T\sin\theta\cdot d\theta = 0. \tag{2}$$

After multiplying Eq. (1) by $\sin\theta$, Eq. (2) by $\cos\theta$, and adding; then multiplying Eq. (1) by $\cos\theta$, Eq. (2) by $-\sin\theta$, and adding; it is found that

$$dT = w\sin\theta\,ds,$$
$$Td\theta = w\cos\theta\,ds.$$

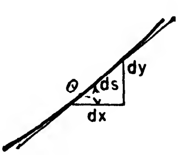


FIG. 101.

But from the differential calculus,

$$dx = \cos\theta\,ds, \qquad dy = \sin\theta\,ds;$$

therefore

$$\frac{dT}{dy} = w, \qquad T\frac{d\theta}{dx} = w. \tag{3}$$

These equations also define the curve, and are sometimes useful in place of Eqs. (189.2) and (189.3).

191. The Curve is a Parabola.—A heavy chain is suspended by its two extremities and forms an arc of a parabola. It is required to find the law of its density.

Let the equation of the parabola be

$$x^2 = 4qy,$$

and therefore

$$\frac{dy}{dx} = \frac{x}{2q} = \frac{W}{H} \qquad \text{by Eq. (189.2).}$$

On differentiating again

$$\frac{H}{2q} = \frac{dW}{dx} = w(s) \cdot \frac{ds}{dx} = w \sec \theta.$$

Hence,

$$w(s) = \frac{H}{2q} \cos \theta;$$

that is, the weight of the chain per unit length at any point, or if preferred, the density of the chain at any point, is proportional to the cosine of the slope of the curve at that point. If the density of the curve is imagined projected upon the x-axis, that is, that the mass in each element ds is distributed uniformly over the corresponding element dx of the x-axis, then the density along the x-axis is constant, as in the suspension bridge (Sec. 186).

192. The Catenary.—The curve which is formed by a uniform heavy chain suspended by its extremities is known as the *catenary*. The weight of the chain per unit length w is constant, and therefore the weight W of an arc of length s is

$$W = ws.$$

If, for the sake of notation, the constant c is defined by the relation

$$H = cw \tag{1}$$

where the constant c has the dimensions of a length, Eq. (189.2) becomes

$$\frac{dy}{dx} = \frac{s}{c}. \tag{2}$$

Since,

$$\left(\frac{dy}{dx}\right)^2 = \left(\frac{ds}{dx}\right)^2 - 1,$$

Eq. (2) can be written

$$\frac{ds}{dx} = \sqrt{1 + \frac{s^2}{c^2}}, \quad \text{or} \quad \frac{ds}{\sqrt{1 + \frac{s^2}{c^2}}} = dx.$$

On integrating, and choosing the constant of integration so that s vanishes with x, there results

$$\frac{s}{c} = \sinh\frac{x}{c}. \tag{3}$$

If this expression for s is substituted in Eq. (2) and a second integration is performed, the equation of the curve of equilibrium, which is called the catenary, is found to be

$$\frac{y}{c} = \cosh\frac{x}{c}, \tag{4}$$

the constant of integration having been taken equal to zero. From the relation

$$\cosh^2\xi - \sinh^2\xi = 1,$$

which holds for every value of ξ, and from Eqs. (3) and (4), it follows that

$$y^2 - c^2 = s^2. \tag{5}$$

It is seen that when x vanishes, y is equal to c, and it is evident that the parameter c is merely a scale-constant. The x-axis of Eq. 4 is sometimes called the *directrix of the catenary.*

193. Tension of the Chain.—It is found by squaring and adding the two equations of Eq. (189.1) that

$$\begin{aligned} T^2 &= W^2 + H^2 = w^2(c^2 + s^2) \\ &= w^2c^2\left(1 + \sinh^2\frac{x}{c}\right) = w^2c^2\cosh^2\frac{x}{c}; \end{aligned}$$

whence

$$T = wy. \tag{1}$$

Equation (1) shows that if a long chain is thrown over two smooth pegs with the two free ends of the chain coming down just to the directrix, the chain will be in equilibrium; and conversely, if the chain is in equilibrium, the free ends come down to the directrix. The proof lies in the fact that the tensions on the two opposite sides of the peg are equal.

If the two points of suspension are on the same level and their common ordinate is y, then the sag of the chain h is $y - c$. Suppose that the two free ends of the chain are very long relative to the sag, so that the chain is very tightly stretched between the pegs. On factoring its left member, Eq. (192.5) can be written

$$y + c = \frac{s^2}{y - c}.$$

Since y and c are nearly equal, this becomes, approximately,

$$c = \frac{s^2}{2h}. \tag{2}$$

Let the distance between the two pegs be l, and let W be the weight of the chain between them. Then, from Eq. (192.1),

$$c = \frac{H}{w},$$

and, approximately, $s = \frac{1}{2} l, \quad W = wl.$

If these values are substituted in Eq. (2), there results the simple approximate equation for the tension in terms of the total weight, length, and sag,

$$H = \frac{Wl}{8h}. \tag{3}$$

The tension at the highest point of the catenary, the pegs or the points of suspension of the chain, can be expanded as a power series in h^2/l^2 which is convergent if h/l is sufficiently small. The result is

$$T = \frac{1}{2} W \coth \frac{l}{2c} = \frac{Wl}{8h}\left(1 + \frac{20}{3}\frac{h^2}{l^2} - \frac{704}{45}\frac{h^4}{l^4} + \cdots\right). \tag{4}$$

It will be noticed that Eq. (3) is merely the first term of this series. The demonstration of Eq. (4), will be left as an exercise.

194. Approximations to the Catenary.—The expansion of the function $\cosh x/c$ as a power series is

$$\cosh \frac{x}{c} = 1 + \frac{1}{2!}\left(\frac{x}{c}\right)^2 + \frac{1}{4!}\left(\frac{x}{c}\right)^4 + \cdots.$$

If x/c is small, the series can be limited to the first two terms, and Eq. (192.4) then becomes

$$y = c + \frac{x^2}{2c};$$

or, on changing the origin to the vertex and replacing the value of c from Eq. (192.1),

$$x^2 = \frac{2H}{w} y. \tag{1}$$

The latus rectum of this parabola is $2H/w$, and the distance to the focus is $H/(2w)$. This approximation is useful for that part of the catenary which is near the vertex.

The function $\cosh x/c$ can also be written

$$\cosh\frac{x}{c} = \frac{e^{\frac{x}{c}} + e^{-\frac{x}{c}}}{2}.$$

Since $e^{-\frac{x}{c}}$ diminishes very rapidly as x increases, for large values of x the catenary closely approximates the exponential curve

$$y = \frac{1}{2}e^{\frac{x}{c}},$$

the approximation improving as x increases.

The expansion for $\sinh x/c$ is

$$\sinh\frac{x}{c} = \left(\frac{x}{c}\right) + \frac{1}{3!}\left(\frac{x}{c}\right)^3 + \frac{1}{5!}\left(\frac{x}{c}\right)^5 + \cdots.$$

Hence, when x/c is small, Eq. (192.3) is approximately

$$s = x + \frac{1}{6}\frac{x^3}{c^2},$$

or

$$s - x = \frac{w^2}{6H^2}x^3.$$

If l is the span of a stretched wire,

$$2s - l = \frac{w^2 l^3}{24H^2}$$

is the increase in the length of the wire necessary on account of the sag.

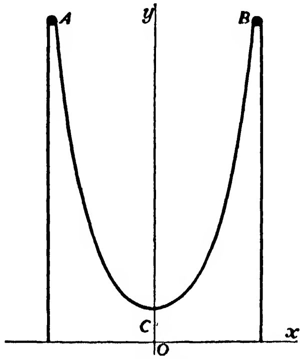


FIG. 102.

195. The Catenary of Uniform Strength.—In the ordinary catenary, in which the chain is uniform, the tension is greatest at the points of suspension, and the chain is most likely to break at one of these points. It is of interest to enquire what must be the cross-section of a wire for which breakage is equally likely everywhere.

If the wire is uniform in every respect save in its cross-section, then the cross-section at any point must be proportional to the tension at that point. If a is the area of the cross-section and T is the tension, then

$$T = ka, \tag{1}$$

where k is a constant factor of proportionality, and a and T are variable from point to point. If σ is the weight of the wire

per unit volume it will be constant, and the value of w the weight of the wire per unit length at the point is

$$w = \sigma \cdot a.$$

If these expressions for T and w are substituted in the second equation of Eq. (190.3), there results

$$\frac{d\theta}{dx} = \frac{\sigma}{k} = \frac{1}{c},$$

and since $k/\sigma = c$ is constant, this is immediately integrable and

$$\theta = \frac{x}{c},$$

the constant of integration being zero. Now

$$\frac{dy}{dx} = \tan\theta = \tan\frac{x}{c};$$

therefore, by a second integration,

$$\frac{y}{c} = \log\sec\frac{x}{c}$$

which is the equation of the curve, the origin being at the lowest point of the curve.

From the first of Eq. (189.1)

$$T = H\sec\theta,$$

and therefore, from Eq. (1),

$$a = \frac{H}{k}\sec\theta.$$

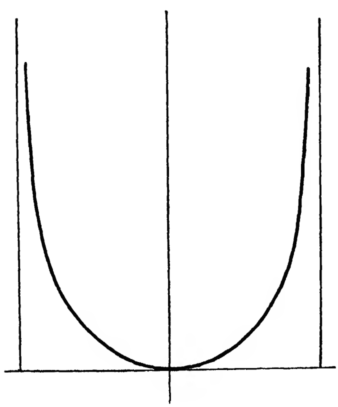


FIG. 103.

Since

$$-\frac{\pi}{2} < \theta < +\frac{\pi}{2},$$

the curve is asymptotic to the lines

$$x = \pm\frac{k\pi}{2\sigma}.$$

The area of the cross-section becomes infinite at these limits, so that in practice these limits could not be attained.

196. The Maximum Span of Any Cable.—If a cable is built out of uniform material, its breaking tension per unit area k

is constant and depends only on the material. By Eq. (190.3),

$$\frac{dx}{d\theta} = \frac{T}{w}$$

for every cable, and the value of θ cannot exceed $\pi/2$, since the cable is flexible. Hence, the maximum span for any cable is

$$\text{maximum span} = \int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} \frac{T}{w}\, d\theta = \frac{1}{\sigma}\int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} \frac{T}{a}\, d\theta,$$

in which T and a are to be regarded as functions of θ. In order that this integral may be a maximum, the ratio T/a must be a maximum, which is the breaking tension per unit area k; that is, the cable must be on the point of breaking everywhere. Hence, the maximum span is

$$\text{maximum span} = \frac{k}{\sigma}\int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} d\theta = \frac{k\pi}{\sigma},$$

which is the maximum span of the catenary of uniform strength; or, rather, it is an upper limit, for this span itself is not attainable.

197. The Maximum Span of a Steel Cable.—For a steel cable for which $k = 130{,}000$ pounds per square inch and $\sigma = 480$ pounds per cubic foot, the maximum span is

$$\frac{\pi \times 130{,}000 \times 144}{480} = 122{,}500 \text{ feet} = 23.2 \text{ miles}.$$

If the factor of safety is taken as 6, the maximum span of safety (tension equal to one-sixth of 130,000) is only one-sixth as much, or 4 miles nearly. If the extreme values of θ are $\pm\pi/6$, then the total span will be $k\pi/(3\sigma)$, and for the above cable and factor of safety, the span is reduced to 4/3 mile. The sag of the cable is,

$$\text{Sag} = \frac{k}{\sigma} \log_e \sec \frac{\pi}{6},$$

which is about 932 feet. The cable is about 7 1/2 per cent thicker at the ends than at the middle.

198. The Elastic Catenary.—In the previous examples, it has been supposed that the wires or cables were inelastic. In the present example, it will be assumed that the wire when not under tension has a uniform cross-section a_0, is homogeneous

throughout, and has a modulus of elasticity λ (Sec. 109). Let the wire of length $2l$ be hanging in equilibrium. Let ds be an element of the wire and a its cross-section; let ds_0 be the length of the same element when it is not stretched; and let σ be the weight per unit volume after stretching and σ_0 the weight per unit volume before stretching.

Since the weight of the element is not altered by stretching,

$$\sigma a ds = \sigma_0 a_0 ds_0.$$

But by Hooke's law (Sec. 109),

$$\frac{ds}{ds_0} = 1 + \frac{T}{\lambda};$$

hence,

$$\sigma a = \frac{\sigma_0 a_0}{1 + \frac{T}{\lambda}}.$$

Since $\sigma \cdot a$ is the weight of the wire per unit length,

$$\sigma a = w, \qquad \sigma_0 a_0 = w_0,$$

it follows from Eq. (190.3), that

$$\frac{dT}{dy} = w = \frac{w_0}{1 + \frac{T}{\lambda}};$$

and therefore

$$\frac{dT}{dx} = \frac{w_0}{1 + \frac{T}{\lambda}} \cdot \frac{dy}{dx}. \tag{1}$$

From Eqs. (189.1) and (189.2), it is found that

$$\frac{dy}{dx} = \sqrt{\frac{T^2}{H^2} - 1}, \tag{2}$$

where H is the horizontal tension of the lowest point of the complete curve. On substituting Eq. (2) in Eq. (1), the equation becomes

$$\left(1 + \frac{T}{\lambda}\right)\frac{dT}{dx} = w_0\sqrt{\frac{T^2}{H^2} - 1},$$

or

$$\frac{1 + \frac{T}{\lambda}}{\sqrt{T^2 - H^2}} dT = \frac{w_0}{H} dx.$$

After integrating and choosing the constant of integration so that

$$x = 0 \text{ when } T = H,$$

it is found that

$$\frac{1}{\lambda}\sqrt{T^2 - H^2} + \cosh^{-1}\frac{T}{H} = \frac{w_0}{H}x. \tag{3}$$

It is very convenient here to introduce a new parameter ω and constants c and e, which are defined by the relations

$$T = H\cosh\omega, \qquad c = \frac{H}{w_0}, \qquad e = \frac{H}{\lambda}; \tag{4}$$

so that Eq. (3) becomes

$$\frac{x}{c} = \omega + e\sinh\omega,$$

and

$$dx = c(1 + e\cosh\omega)d\omega. \tag{5}$$

On substituting Eqs. (4) and (5) in Eq. (2), there results

$$\frac{1}{c}dy = (1 + e\cosh\omega)\sinh\omega\, d\omega,$$

and therefore, by integration,

$$\frac{y}{c} = \cosh\omega + \frac{1}{2}e\cosh^2\omega,$$

the constant of integration being taken equal to zero.

The parametric equations of the elastic catenary are, therefore,

$$\left.\begin{aligned} x &= c(\omega + e\sinh\omega), \\ y &= c\left(\cosh\omega + \frac{1}{2}e\cosh^2\omega\right). \end{aligned}\right\} \tag{6}$$

For $e = 0$, or what is the same thing, $\lambda = \infty$, the elastic catenary reduces to the ordinary catenary as, of course, it should. Since the ordinary catenary depends upon the single constant c, all ordinary catenaries have the same shape. They differ from one another only in size. Elastic catenaries depend not only upon the constant c, but also upon the constant e which can be regarded as defining the shape. Hence, elastic catenaries differ from one another not only in size but also in shape.

199. The Stretch of the Wire.—The length of the wire in its position of equilibrium measured from the lowest point of the curve to the point of suspension, for which $\omega = \omega_1$, is

$$s_1 = \int_0^{s_1} ds = c\int_0^{\omega_1}(1 + e\cosh\omega)\cosh\omega\, d\omega,$$

or

$$s_1 = c\left[\sinh\omega_1 + \frac{1}{2}e\left(\omega_1 + \frac{1}{2}\sinh 2\omega_1\right)\right].$$

For the unstretched wire,

$$s_0 = \int_0^{s_0} ds_0 = \int_0^{s_1}\frac{ds_0}{ds}ds = c\int_0^{\omega_1}\frac{1 + e\cosh\omega}{1 + \frac{T}{\lambda}}\cosh\omega\, d\omega$$

$$= c\int_0^{\omega_1}\cosh\omega\, d\omega = c\sinh\omega_1.$$

Hence, the stretch of the wire is

$$\text{stretch} = s_1 - s_0 = \frac{1}{2}ce(\omega_1 + \frac{1}{2}\sinh 2\omega_1). \tag{1}$$

The sag of the wire is

$$\text{sag} = y(\omega_1) - y(0) = c\left[(\cosh\omega_1 - 1) + \frac{1}{2}e\sinh^2\omega_1\right] \tag{2}$$

$$= \sqrt{s_0^2 + c^2} - c + \frac{e}{2c}s_0^2.$$

If the stretch in the wire is small, as in a steel wire, Eq. (2) can be expanded as a power series in the parameter e. The sag for the inelastic wire, using the same points of suspension, distance apart $2a$, and the same length of wire s_0 is

$$\text{sag}_0 = \sqrt{s_0^2 + c_0^2} - c_0$$

where c_0 is the scale constant for the inelastic wire. It is found then for the elastic wire that

$$\text{sag} = \text{sag}_0 + ce\left[\frac{\cosh\omega_1 - 1}{\frac{a}{s_0}\cosh\omega_1 - 1} + \frac{s_0^2}{2c^2}\right] + \cdots,$$

so that the extra sag due to stretch is

$$\text{sag due to stretch} = \left[\frac{\cosh\omega_1 - 1}{\frac{a}{s_0}\cosh\omega_1 - 1} + \frac{s_0^2}{2c^2}\right]ce + \cdots. \tag{3}$$

200. Numerical Examples.—If the given length of the unstretched wire is $2s_0$ and the horizontal distance between the two points of suspension is $2a$, then

$$a = c_1[\omega_1 + e\sinh\omega_1] = c_1\left[\omega_1 + \frac{w_0}{\lambda}s_0\right],$$

$$s_0 = c_1\sinh\omega_1.$$

The value of ω_1 is obtained from the equation

$$\frac{s_0}{a} = \frac{\sinh \omega_1}{\omega_1 + \dfrac{w_0}{\lambda} s_0},$$

a fairly simple matter with a table of the hyperbolic functions. Then c_1 is derived from the equation

$$c_1 = \frac{s_0}{\sinh \omega_1}.$$

Consider, for example, a steel wire 1/8 in. in diameter for which

$$w = \frac{1}{24} \text{ lb.,} \qquad \lambda = 4.3 \times 10^5 \text{ lb.}$$

The wire is 2100 ft. long and the distance between the points of suspension is 2000 ft. What is the sag?
The value of ω_1 is given by the equation

$$1.05 = \frac{\sinh \omega_1}{\omega_1 + 0.969 \times 10^{-7} s_0} = \frac{\sinh \omega_1}{\omega_1} \text{ sensibly.}$$

Hence,

$$\omega_1 = 0.5450, \qquad \sinh \omega_1 = 0.5724, \qquad c_1 = 1834 \text{ ft.}$$

and the sag is 279.6 ft.

It will be observed that these figures are not sufficiently accurate to make the sag due to the stretch sensible, but it is easily computed by means of Eq. (199.3), and is found to be about 10.2 inches.

201. A Rubber String.—Consider a rubber string 200 feet long suspended from two points 150 feet apart. The value of w is

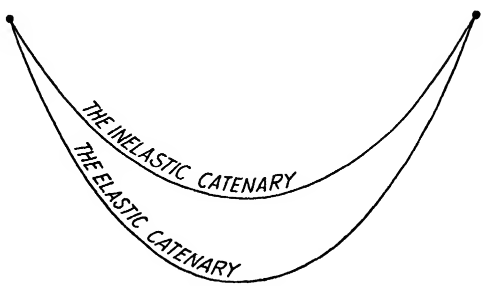


FIG. 104.

one-twelfth pound and λ is 25 pounds. From the equations of the preceding section, it is found that

$$\omega_1 = 1.742, \qquad c_1 = 36.14;$$

and, for an inelastic string under the same circumstances,

$$\omega_0 = 1.351, \qquad c_0 = 55.51.$$

The two curves are drawn in Fig. 104 to illustrate the sag due to stretch. The two sags are 86.8 and 58.9 feet, respectively. The sag due to stretch is 27.9 feet, while Eq. (199.3) gives 36.6 feet, showing that higher terms of the expansion are necessary when the stretch is so large. The first term is sufficient only when the stretch is very small.

202. Catenaries in General.—Imagine a perfectly flexible chain with infinitely small links fastened at its two ends in equilibrium in an arbitrary field of force. Let ds be any one of its links which is acted upon by the tensions of the two adjoining links $\mathbf{T}_1$ and $\mathbf{T}_2$ and by a force $\mathbf{f}$ which is due to the field. Let σ be the mass of the chain per unit length, and let $\mathbf{F}$ be the force which is due to the field acting upon a unit mass. Then the mass of the link is

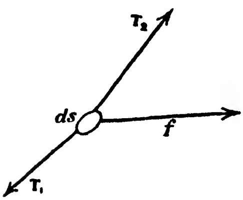


FIG. 105.

$$dm = \sigma ds$$

and

$$\mathbf{f} = \mathbf{F}\sigma ds.$$

Let the forces be resolved along the x-, y-, and z-axis, and let X, Y, and Z be the magnitudes of the components of $\mathbf{F}$. Then

$$(T_2 - T_1)_x = dT_x = -X\sigma ds,$$
$$(T_2 - T_1)_y = dT_y = -Y\sigma ds,$$
$$(T_2 - T_1)_z = dT_z = -Z\sigma ds,$$

where T_x, T_y, and T_z are the components of the tension T acting at ds. Therefore, by projection upon the x-, y-, and z-axes,

$$T_x = T\cdot\frac{dx}{ds}, \qquad T_y = T\cdot\frac{dy}{ds}, \qquad T_z = T\cdot\frac{dz}{ds}.$$

The differential equations for the catenary become, therefore,

$$\left.\begin{aligned} \frac{d}{ds}\left(T\frac{dx}{ds}\right) &= -\sigma X, \\ \frac{d}{ds}\left(T\frac{dy}{ds}\right) &= -\sigma Y, \\ \frac{d}{ds}\left(T\frac{dz}{ds}\right) &= -\sigma Z. \end{aligned}\right\} \qquad (1)$$

203. Hypothesis—There Exists a Force Function.—Suppose that there exists a force function V (Sec. 64), so that

$$X = \frac{\partial V}{\partial x}, \qquad Y = \frac{\partial V}{\partial y}, \qquad Z = \frac{\partial V}{\partial z}.$$

On multiplying the first equation Eq. (202.1) by dx, the second by dy, the third by dz, and adding, there results

$$\begin{aligned}\frac{d}{ds}\left(T\frac{dx}{ds}\right)dx + \frac{d}{ds}\left(T\frac{dy}{ds}\right)dy + \frac{d}{ds}\left(T\frac{dz}{ds}\right)dz \\ = -\sigma\left(\frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz\right) \\ = -\sigma dV.\end{aligned}$$

If the left member of this equation is developed, it becomes

$$\frac{dT}{ds}\left(\frac{(dx)^2+(dy)^2+(dz)^2}{ds}\right)+T\left(\frac{dx}{ds}\frac{d^2x}{ds^2}+\frac{dy}{ds}\frac{d^2y}{ds^2}+\frac{dz}{ds}\frac{d^2z}{ds^2}\right)ds = -\sigma dV;$$

and since

$$(dx)^2 + (dy)^2 + (dz)^2 = (ds)^2,$$

and

$$\begin{aligned}\frac{dx}{ds}\frac{d^2x}{ds^2} + \frac{dy}{ds}\frac{d^2y}{ds^2} + \frac{dz}{ds}\frac{d^2z}{ds^2} &= \frac{1}{2}\frac{d}{ds}\left(\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 + \left(\frac{dz}{ds}\right)^2\right) \\ &= \frac{1}{2}\frac{d}{ds}(1) = 0,\end{aligned}$$

it reduces to

$$dT = -\sigma dV.$$

For a uniform chain, σ is constant and the integral is

$$T = \sigma(V_0 - V). \tag{1}$$

The potential energy of an element of the chain

$$dm = \sigma ds$$

with respect to the given field of force is $-V\sigma ds$. The potential energy of the entire chain is, therefore,

$$-\int_{s_1}^{s_2} V\sigma ds,$$

the integral being taken along the catenary itself.

Problems XVII

1. A heavy chain hangs over two smooth pegs on the same level and at a distance a apart with the two ends hanging freely and the central part in the form of a catenary. Show that if the chain is in equilibrium and its length is l

$$l \geqq ae \ (e = \text{naperian base}).$$

2. An elastic string, of natural length na, has n equal weights w attached to it at distances a, $2a$, $3a$, $\cdots$ from the end by which it is suspended. Find the amount of stretch and its potential energy. *Ans.* Stretch $= n(n+1)\,wa/(2\lambda)$; potential energy $= n(n+1)(2n+1)w^2a/(12\lambda)$.

3. An elastic string 4 ft. long has two weights of 2 lb. each attached to it, one at an end and one in the middle. The elasticity of the string is such that a 1-lb. weight attached to its end lengthens the string 4/7 ft. The string with its weights is lifted from the ground by its unweighted end. What is the least amount of work which will lift both weights from the ground? *Ans.* 8 ft.-lb.

4. A chain of weight C is suspended by its extremities from two points on the same level and a weight W is attached to its lowest point. If α_1 and α_2 are the angles with the horizontal made by the tangents to the curve at W and the points of suspension, respectively, show that

$$\frac{\tan\alpha_2}{\tan\alpha_1} = 1 + \frac{C}{W}.$$

5. A chain of length l is hung from two points on different levels, and at these points makes angles α and β with the horizontal. Show that the difference in level of the two points is

$$l\,\frac{\sin\frac{1}{2}(\alpha+\beta)}{\cos\frac{1}{2}(\alpha-\beta)}.$$

6. A chain 110 ft. long is suspended from two points in the same horizontal line 108 ft. apart. Show that the tension at its lowest point is 1.477 times the weight of the chain.

7. A chain 10 ft. long and weighing 30 lb., suspended from two points in the same horizontal line, has a sag of 4 ft. Find the tension at its lowest point and the angle which the tangents to the curve at the points of suspension make with the horizontal. *Ans.* Tension = 3 3/8 lb., angle = 77° 19′.

8. An elastic string of natural length l and weight w per unit length is attached to two fixed points in the same vertical line at a distance b apart, so that its stretch is greater than its own weight would produce. If λ is the modulus of elasticity of the string, show that the total displacement of a point whose natural distance from the top is x, is

$$\frac{wx}{2\lambda}(l-x) + \frac{bx}{l}.$$

9. A funicular polygon is formed by attaching n equal weights equally spaced to a light string of length l. Show that if n is indefinitely increased, but the sum of the weights is fixed, the funicular polygon has the catenary as a limit.

10. A heavy non-uniform chain suspended from two points on the same level forms a semi-circle. If y is the distance of a link of the chain

below the line of suspension and r is the radius of the circle, show that the density of the chain is proportional to

$$\frac{r}{(r - y)^2}.$$

11. Show that the work done in stretching the cord in the elastic catenary is

$$W = 2w_0c^2e\left(\sinh \omega + \frac{1}{3} \sinh^3 \omega\right).$$

12. If the middle point of the chain in Fig. 102 is constrained to move along the y-axis, show that the equilibrium is stable or unstable, according as $a \gtrless c$.

13. Prove that the configuration of Fig. 102 is also a configuration of equilibrium for the elastic catenary.

14. The extremities of a chain are fastened to two fixed points A and C in the same horizontal line. Midway between A and C, at B, there is a smooth peg over which the chain passes. Show that the symmetrical configuration of equilibrium is unstable or stable according to whether there exists or does not exist an unsymmetrical configuration of equilibrium.

15. The load supported by a chain is $ax^2 + b$ per horizontal foot. What is the equation of its curve? *Ans.* $Hy = ax^4/12 + bx^2/2$.

16. A chain hangs freely from two points on the same level. At equal distances from its two ends, a second chain is attached to the first. Required: the configuration of equilibrium.

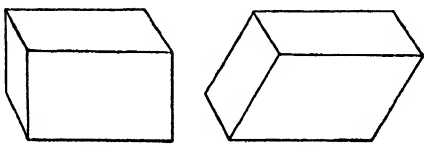
17. A smooth heavy wire is suspended from two pegs in a horizontal line, at a distance $2a$ apart. It passes through a small ring which is attached by a light wire to a point halfway between the pegs. The length of the fine wire is less than half the distance between the pegs. Required: the configuration of equilibrium.

III. ELASTIC SOLIDS

204. Deformation.—If the shape of a solid body is altered the body is said to be *deformed.* Different substances react quite differently to slight deformations. If a lump of putty is deformed, it will stay deformed, showing no tendency to return to its initial state. Such a substance is said to be *plastic.* If, however, a piece of spring steel is slightly deformed and then released, it will return promptly to its initial state. Such a substance is said to be *elastic.* In the remainder of the present chapter, solids that are elastic will be discussed.

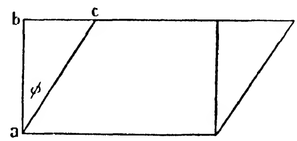
205. Strains.—Any definite alteration of the form or dimensions of an elastic body is called a *strain.* The simplest types of strains are a *homogeneous elongation* (or contraction), as of a rod,

and a *simple shearing strain* such as that in which a right parallelepiped is converted into an oblique parallelepiped, the base and altitude, and therefore the volume, remaining unaltered. The simple shearing strain is well illustrated by a deck of ordinary playing cards which is deformed from its initial figure of a right parallelepiped by sliding each card in its own plane parallel to the edge of the deck by an amount which is proportional to its distance from the bottom card.



(*a*) (*b*)

FIG. 106.



(*c*)

FIG. 107.

If the natural length of a rod is l and its strained length is $l + s$, the measure of the strain is s/l, which is the strain per unit length, and is constant throughout the rod if the strain is homogeneous, that is, all parts of the rod are strained alike. If the rod is lengthened, s is positive; if it is shortened, s is negative.

The measure of a shearing strain, or more simply a shear, is the tangent of the angle ($\overline{bc}/\overline{ab}$, Fig. 107, that is, the strain per unit length), through which the edge of the parallelepiped is rotated; or, since the angle is usually very small, it is the radian measure of the angle φ itself.

206. Stress.—Any force acting between the parts of a solid body is called a *stress*. Very little is known about the forces that are acting between the parts of a solid body except that, when the body is not acted upon by any external forces, the internal forces are in equilibrium among themselves. When external forces are applied to the body the internal forces are altered and therefore, in general, the shape of the body is altered. It is the alteration of the internal forces, brought about by the application of external forces, with which this discussion is chiefly concerned. Thus, if a rod is subjected to equal and opposite pulls at its two ends, the stress induced in the rod is called a *tension*. If the pulls are replaced by pushes the induced stress is a *thrust*, so that, from a mathematical point of view, a thrust is merely a negative tension.

207. Shearing Stress.—A shearing stress between two particles tends to produce motion in a line which is perpendicular to the line joining the particles. Thus, if a heavy box is resting on a table, the mutual action of the box and the table *across* the plane which separates them is a thrust. But if the box is given a slight push or pull parallel to the table, without motion occurring, the induced stress *in* the plane which separates them is a shearing stress. It is evident that shearing stresses produce shearing strains.

FIG. 108.

The diagram which represents a shearing stress is like the diagram of a couple with a very short arm. When the sign of a shearing stress is of importance, it will be regarded as positive when the moment of the couple would be positive and negative when the moment of the couple would be negative.

208. Generalized Hooke's Law.—Many careful experiments on elastic bodies show that Hooke's law, originally stated for elastic strings, can be extended to all kinds of stresses and strains provided they are not too large. Thus, *the stress producing any strain is proportional to the strain produced.* In the form of an equation

$$\frac{\text{Stress per unit area}}{\text{Strain per unit length}} = \text{constant}$$

for any given type of strain and any given substance. If the strain is a simple elongation, this constant is called *Young's modulus*, and is denoted by the letter M. If the strain is a shear, this constant is called the *modulus of rigidity* and is denoted by the letter n. Approximate values of these moduli, measured in pounds per square inch,[1] are given in the following table for a few common substances:

Substance	Density	n = modulus of rigidity	M = Young's modulus	Breaking tension
Flint glass	2.94	3.45×10^6	8.5×10^6	1×10^4
Brass	8.47	5.5×10^6	15×10^6	1×10^4
Steel	7.85	11.9×10^6	35×10^6	10×10^4
Wrought iron	7.68	11.2×10^6	28.5×10^6	7×10^4
Cast iron	7.24	7.7×10^6	19.6×10^6	1.6×10^4
Copper	8.84	6.5×10^6	17.5×10^6	6×10^4

[1] To convert from pounds per square inch to dynes per square centimeter, multiply by 68,900.

209. Elastic Rods.—In the case of an elastic rod, the stress per unit area is the tension in the rod T divided by the area of the cross-section a. If s is the stretch of the rod (*i.e.*, the extended length minus the natural length) and l is the natural length, the above equation gives

$$\frac{T}{a} = M\frac{s}{l}, \qquad \text{or} \qquad T = Ma\frac{s}{l}.$$

The constant Ma depends not only upon the substance but also upon the body. It has heretofore been denoted by the letter λ and called the modulus of elasticity.

210. Application to Overrigid Frameworks.—If a framework is overrigid and the rods are regarded as rigid, the stresses in the rods cannot be determined. If the rods are regarded as elastic, the changes in the stresses due to the action of external forces can be determined. Since the framework is overrigid, there may exist stresses in the rods without any outside forces acting. These stresses are, in general, unknown, and remain so. It is only the changes in the stresses due to outside forces that are sought.

211. First Example.—Let there be given a freely jointed framework in the form of a square with the two diagonals, the rods being all alike and of the same material. Let the framework rest upon a table with one diagonal vertical, and support a weight W. Required: the changes in the stresses of the rods due to the weight W.

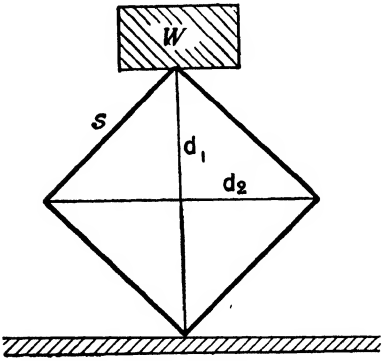


FIG. 109.

Let s be the length of each of the sides and $d = d_1 = d_2$ be the length of the diagonals. Then

$$d = s\sqrt{2}.$$

It is evident from symmetry that the stresses in the sides are all alike and, therefore, these changes in length are all alike. Let the stretches in these rods be denoted by σ; let d_1 be the vertical diagonal and δ_1 its stretch; and let d_2 be the horizontal

rod and δ_2 its stretch. Let S, D_1, and D_2 be the stresses in the sides and diagonals, respectively. Then

$$S = \lambda\frac{\sigma}{s}, \qquad D_1 = \lambda\frac{\delta_1}{d_1}, \qquad D_2 = \lambda\frac{\delta_2}{d_2}.$$

After the load W is placed upon the framework, the diagonals will still be perpendicular to each other, since the square has become a rhomb. Hence,

$$\left(\frac{d_1}{2}\right)^2 + \left(\frac{d_2}{2}\right)^2 = s^2, \qquad \left(\frac{d_1 + \delta_1}{2}\right)^2 + \left(\frac{d_2 + \delta_2}{2}\right)^2 = (s + \sigma)^2.$$

On neglecting the squares of the small quantities and bearing in mind that

$$d_1 = d_2 = s\sqrt{2},$$

it is found that

$$\delta_1 + \delta_2 = 2\sqrt{2}\sigma;$$

and on multiplying through by

$$\frac{\lambda}{d} = \frac{\lambda}{s\sqrt{2}},$$

it becomes

$$D_1 + D_2 = 2S.$$

Since the hinge at W is in equilibrium,

$$W + S\sqrt{2} + D_1 = 0;$$

and since the hinges at the ends of the horizontal diagonal d_2 are in equilibrium

$$S\sqrt{2} + D_2 = 0.$$

The solution of these last three equations gives

$$S = -\frac{1}{2}(\sqrt{2} - 1)W,$$

$$D_1 = -\frac{1}{2}\sqrt{2}W,$$

$$D_2 = +\frac{1}{2}(2 - \sqrt{2})W.$$

There is a tension in d_2, and thrusts in d_1 and all of the sides.

212. Second Example.—Two heavy bars are connected by three elastic rods, not necessarily alike but of the same length s. The distances from the middle rod to the end rods are a and b, respectively. A pull $\mathbf{F}$, perpendicular to the bars, is applied at

a distance c from one end of the bars. Determine the stresses in the rods.

Let the moduli of elasticity of the three rods be λ_1, λ_2, and λ_3, respectively, and let their stretches be σ_1, σ_2, and σ_3. Since the heavy bars are essentially rigid,

$$\frac{\sigma_1 - \sigma_2}{a} = \frac{\sigma_2 - \sigma_3}{b}. \qquad (1)$$

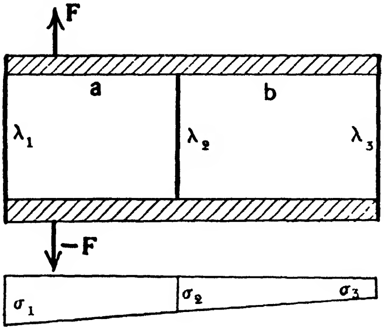


FIG. 110.

If S_1, S_2, and S_3 are the stresses in the rods,

$$S_1 = \lambda_1\frac{\sigma_1}{s}, \qquad S_2 = \lambda_2\frac{\sigma_2}{s}, \qquad (2)$$

$$S_3 = \lambda_3\frac{\sigma_3}{s}.$$

Since the upper bar is in equilibrium, the resolution of the forces parallel to the direction of the rods and the principle of moments give the equations

$$S_1 + S_2 + S_3 = F, \qquad (3)$$
$$aS_2 + (a + b)S_3 = cF. \qquad (4)$$

The solution of Eqs. (1) to (4), for S_1, S_2, and S_3 gives

$$S_1 = \frac{a(a - c)\lambda_2 + (a + b)(a + b - c)\lambda_3}{a^2\lambda_1\lambda_2 + b^2\lambda_2\lambda_3 + (a + b)^2\lambda_1\lambda_3}\lambda_1 F,$$
$$S_2 = \frac{ac\lambda_1 + b(a + b - c)\lambda_3}{a^2\lambda_1\lambda_2 + b^2\lambda_2\lambda_3 + (a + b)^2\lambda_1\lambda_3}\lambda_2 F,$$
$$S_3 = \frac{(a + b)c\lambda_1 - b(a - c)\lambda_2}{a^2\lambda_1\lambda_2 + b^2\lambda_2\lambda_3 + (a + b)^2\lambda_1\lambda_3}\lambda_3 F.$$

For the particular case in which

$$\lambda_1 = \lambda_2 = \lambda_3, \qquad a = b, \qquad c = 0,$$

the stresses are

$$S_1 = \frac{5}{6}F, \qquad S_2 = \frac{1}{3}F, \qquad S_3 = -\frac{1}{6}F.$$

213. The Shear of a Snubbing Post.—A snubbing post is a post planted firmly in the ground, in a dock or on the deck of a ship around which a rope is thrown for the purpose of controlling the motion of heavy objects, such as a horse, a barge, or a ship.

Suppose the snubbing post is a column of steel 8 inches in diameter and a force of 120 tons is applied by means of a rope 30 inches from the ground. How much does the shaft yield to the shearing stress to which it is subject?

It is clear that the shearing stress is constant throughout the post, for, on considering any plane section of the post parallel to the ground, as at B (Fig. 111), it is apparent that the lower part of the post BC exerts a force of 120 tons on the part AB opposite in direction to AR, since the upper part of the post AB is in equilibrium, and therefore *in* the plane at B. There are, of course, other forces acting *across* the plane at B which is equivalent to a couple acting on AB and which balances the couple due to the shearing stresses. These forces are effective in bending the post. They are neglected and only the shear considered.

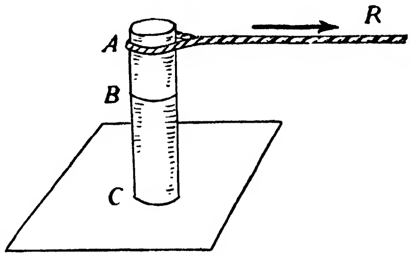


FIG. 111.

Since the shearing stress is constant throughout the post the shearing strain also is constant. Let the angle of shear be φ. The shearing stress per unit area is

$$\frac{120 \times 2000}{16\pi} \text{ pounds per square inch.}$$

The strain is $\tan \varphi$, and the modulus of rigidity n for steel is 11.9×10^6 pounds per square inch. Hence,

$$\frac{120 \times 2000}{16\pi} = 11.9 \times 10^6 \times \tan \varphi,$$

and

$$\tan \varphi = 0.000401.$$

The deflection of the post due to shear is

$$30 \tan \varphi = 0.012 \text{ inches.}$$

214. The Shear of the Frustum of a Cone.—The frustum of a cone would make a poor snubbing post, but it will serve well enough as an illustrative example. Let its height be h, the radius of its top a, and of its base b. Let R be the magnitude of a horizontal force applied at the top, and y the deflection of the cone at the height x due to shear.

The total shearing stress in the horizontal plane at x is, of course, R; but the shearing stress *per unit area* is

$$\frac{R}{\pi\left[b - (b - a)\frac{x}{h}\right]^2},$$

which is variable. The strain $\tan\varphi$ at the height x is dy/dx. Hence,

$$n\frac{dy}{dx} = \frac{R}{\pi\left[b - (b - a)\frac{x}{h}\right]^2}.$$

After integrating and choosing the constant of integration so that y vanishes with x, there results

$$y = \frac{Rh}{n\pi(b - a)}\left[\frac{1}{b - (b - a)\frac{x}{h}} - \frac{1}{b}\right].$$

The deflection at the top is obtained by setting

$$x = h$$

in this equation. Its value is

$$D = \frac{Rh}{n\pi ab}.$$

It is seen from this equation that the deflection for a cone, due to shearing stress, is the same as that of a cylinder of the same height and radius

$$r = \sqrt{ab}.$$

215. Torsion of a Uniform Cylindrical Shaft.—Torsion is a strain of a solid body in which planes which are perpendicular to a certain axis are rotated relatively to one another about that axis. Imagine that the lower end of a uniform cylindrical shaft is kept fixed while the upper end is twisted through an angle θ by a couple of moment C. It is desired to determine the relation between θ and C in terms of the known constants of the shaft.

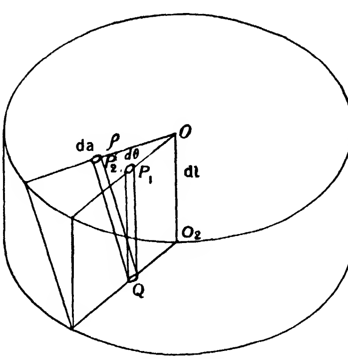


FIG. 112.

Let r be the radius of the cylinder, l its length, and n the modulus of rigidity of the material of which it is composed. Consider a thin section of the shaft of thickness dl formed by two planes perpendicular to the axis. When the shaft is strained the upper plane of this section is rotated, relatively to the lower, through an angle $d\theta$, the axis of rotation

being the axis of the cylinder by virtue of symmetry. Let P_1Q be a cylindrical element of the volume of the section parallel to the axis before it is strained, and let P_2Q be the same element after the strain has occurred. Let da be the area of the cross-section of the cylindrical element made by the upper plane. If ρ is the distance of this element from the axis of rotation, $\rho d\theta$ is the displacement of da from its unstrained position. Since the length of the element before it is strained is dl, the measure of the shearing strain of this element is $\rho d\theta/dl$.

The stress per unit area is, by Hooke's law, $n\rho d\theta/dl$; therefore the force acting on the element da perpendicular to the line joining it to the axis of rotation, and the moment of this force with respect to the axis of rotation, respectively, are

$$n\rho\frac{d\theta}{dl}da, \qquad \text{and} \qquad n\frac{d\theta}{dl}\rho^2 da.$$

The sum of the moments taken over the entire area of the circular cross-section is the moment of the couple, or the torque, which acts upon the upper surface of the section. That is,

$$C = n\frac{d\theta}{dl}\int \rho^2 da.$$

The integral in the right member of this expression is the moment of inertia of the circular area which, by Routh's rule (Sec. 99), is $\pi r^4/2$. Hence, the moment of the couple is

$$C = \frac{1}{2}\pi n r^4 \frac{d\theta}{dl}. \tag{1}$$

There is nothing to distinguish one cross-section of the shaft from another, except its position, and therefore the torsion is constant throughout the shaft. That is,

$$\frac{d\theta}{dl} = \frac{\theta}{l};$$

and accordingly,

$$C = \frac{\pi n r^4}{2l}\cdot\theta.$$

The torsion, therefore, is proportional to the torque.

Since the work done by a couple of moment C in turning through an angle $d\theta$ is

$$dW = Cd\theta,$$

the work done in turning through the finite angle θ is

$$W = \frac{\pi n r^4}{2l}\int_0^\theta \theta d\theta = \frac{\pi n r^4}{4l}\theta^2.$$

This is, therefore, the potential energy of the twist.

If the cross-sections of the shaft are not circles, they are not only rotated but are also distorted, and the arguments given above are not sufficient to determine exactly the corresponding relationships; but they will give results which are quite close for regular polygons of six or more sides. It was found by St. Venant that the couple required to produce a given twist for a rod of square cross-section is only 0.84 of that which is obtained by the above method. The sharper the corners the poorer is the approximation.

216. The Spiral Spring.—An elastic wire which has been shaped by winding on a cylindrical core is commonly called a spiral spring, although more accurately it is a helical spring. It will be supposed that the ends of the wire have been bent so as to lie on the axis of the cylinder. It was pointed out by Binet in 1814, and verified by very careful experiments by J. Thompson in 1848, that such a spring acts chiefly by torsion of the wire, the flexure of the wire being unimportant. Assuming that this is true, the extension of the spring for a given pull can be computed in terms of the constants of the wire. Figure 113 is such a spring constructed of two wires for the purpose of symmetry. It is, of course, just twice as strong as it would be if one of the wires were removed, the other remaining unaltered.

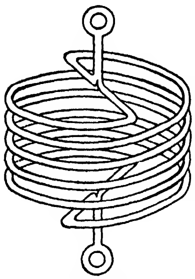


Fig. 113.

Let r be the radius of the wire, a the radius of the coils, l the length of the coiled wire, n its modulus of rigidity, and θ the angle of twist in the wire when the spring is under tension. It will be assumed that the coils lie in planes which are perpendicular to the axis of the cylinder, although this is not quite true. Let p be any point on the coils of the spring when it is under a tension due to two equal and opposite pulls of magnitude P (or $2P$, if it is a double spring as in Fig. 113) applied at the ends of the spring on the axis of the cylinder.

Consider a cross-section of the wire at p formed by a plane which passes through the axis of the cylinder. The action of the

portion of the spring on one side of this plane upon the portion on the other side of the plane is equivalent to a force P parallel to the axis and a couple. Since each portion of the spring is in equilibrium, the sum of the forces parallel to the axis of the cylinder is zero and, therefore, the magnitude of the force is P; and, since the two forces P acting on the two ends of either portion have lines of action which are parallel but at a distance a apart, these two forces produce a couple of moment aP which is balanced by the couple acting at p. The moment of the couple acting at p is, therefore, aP also. By Eq. (215.1), therefore,

$$\frac{1}{2}\pi n r^4 \frac{d\theta}{dl} = aP = \frac{1}{2}\pi n r^4 \frac{\theta}{l},$$

and the work done in twisting the entire length l of the wire is

$$W = \frac{\pi n r^4}{4l}\theta^2 = \frac{la^2P^2}{\pi n r^4}.$$

If x is the amount by which the spring is extended, the work done by the force P in extending the spring is

$$W = \int_0^x P dx$$

and, since the work done by flexure is negligible, all of the work going into the twist, it follows that

$$\int_0^x P dx = \frac{la^2}{\pi n r^4} P^2.$$

On differentiating, and then removing a factor P, there results

$$\frac{dP}{dx} = \frac{\pi n r^4}{2la^2},$$

which is constant. Therefore,

$$x = \frac{2la^2}{\pi n r^4} P = \frac{4c}{nr}\left(\frac{a}{r}\right)^3 P,$$

where c is the number of the coils and, approximately,

$$l = 2\pi c a.$$

This is the relation between the force P and the extension of the spring which was sought.

217. Flexure.—Flexure is a strain in which certain planes within the body are bent into cylindrical or conical surfaces.

Suppose that a rod or a beam is given which, whatever its cross-section may be, is uniform throughout its length. The forces, which are acting on its ends only, can be thought of as tensions or thrusts, shears or couples, or any combination of these; but a beam in equilibrium will be considered first upon which the only forces acting are couples at its extremities. Since the beam is in equilibrium, the two couples are evidently equal and opposite.

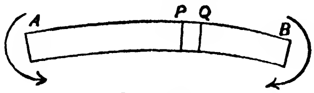


FIG. 114.

Consider a cross-section at any point P. The part of the beam AP, also, is in equilibrium, and therefore the action of the part PB upon the part AP is a couple which is equal to the couple at B. The same is true of the cross-section at any other point Q. Hence, every section of the beam included between two cross-sections is acted upon, by the parts outside of the section, by equal and opposite couples. However any thin section PQ may be deformed, all such sections are deformed alike, and the curvature of the beam is constant. The beam, therefore, is bent into the arc of a circle.

218. The Bending Moment.—The radius of the circle in terms of the applied couples and the constants of the beam will now be determined.

Let $A_1B_1C_1D_1$ be a side view of the beam in its strained position and $A_1A_2D_2D_1$ an end view. It is evident that the beam is

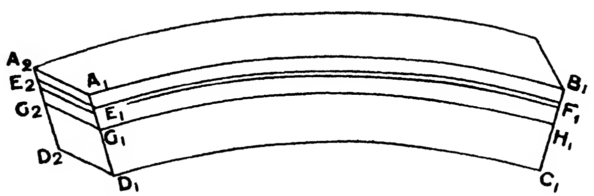


FIG. 115.

stretched along the top plane A_1B_1 and compressed along the bottom plane D_1C_1. There exists, therefore, a plane G_1H_1 which is neither stretched nor compressed, and which is called the *neutral plane*. In the strained position of the beam this plane is, of course, cylindrical with a radius R. If the beam subtends an angle θ at the axis of the cylinder and l is the length of the beam, then

$$l = R\theta = \text{arc } G_1H_1.$$

Let E_1F_1 be a thin plane section of the beam parallel to G_1H_1, and at a distance y from it; y is to be regarded as positive on one side of the plane and negative on the other side. The length of E_1F_1 in its strained position is

$$\text{arc } E_1F_1 = (R + y)\theta.$$

Its stretch is $y\theta$, and its strain per unit length is y/R.

Let da be the area of the cross-section of this thin strip E_1E_2. Then the tension necessary to stretch this strip is (Sec. 209)

$$M\frac{y}{R}da;$$

where M is Young's modulus for the material of the beam. The total force acting across the end is, therefore,

$$\frac{M}{R}\int yda;$$

and, since the force acting is a couple, this integral must be zero. The line G_1G_2, therefore, passes through the center of gravity of the end of the beam $A_1A_2D_2D_1$.

The moment of the tension acting along E_1E_2 with respect to the line G_1G_2 is

$$y \cdot M\frac{y}{R}da = \frac{M}{R}y^2da.$$

Consequently, the moment of the couple C which is acting upon the end of the beam is

$$C = \frac{M}{R}\int y^2da = \frac{MI}{R},$$

where I is the moment of inertia of the cross-section of the beam with respect to the line G_1G_2 through the center of gravity of the cross-section. The couple C is called the *bending moment.* This equation gives the relation between the bending moment and the radius of the cylinder.

219. The Work Done in Bending.—If the beam is naturally straight and it is acted upon only by opposing couples on its ends, the beam is bent into the arc of a circle of radius R.

Let l be the length of the beam, θ the angle through which the end of the beam is turned, and R the radius of the circle into which the beam is bent. The work done by the couple is

$$W = \int_0^\theta Cd\theta = MI\int_0^\theta \frac{d\theta}{R},$$

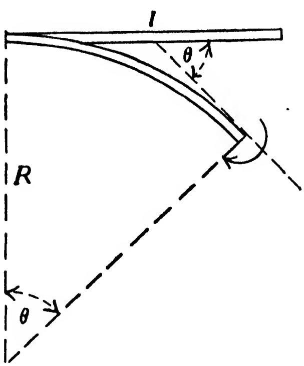


FIG. 116.

or, since

$$R\theta = l,$$

$$W = \frac{MI}{l}\int_0^\theta \theta d\theta = \frac{MI}{2l}\theta^2$$

$$= \frac{MIl}{2R^2},$$

which is the potential energy of the bent beam.

The work done in bending the beam from a circle of radius R_1 to a circle of radius R_2 is

$$W = \frac{1}{2}MIl\left(\frac{1}{R_2^2} - \frac{1}{R_1^2}\right).$$

220. The Beam is Also under Tension.—If, in addition to the couple in Sec. 218, there is also a tension acting across the end of the beam, the sum of the tensions is not zero, and therefore

$$\int y da = y_1 \cdot a,$$

where a is the area of the end of the beam, and y_1 is a distance perpendicular to the line G_1G_2 through the center of gravity of the area. The point of application of the single force **T** which is equivalent to the sum of all of the tensions, lies on a line which is parallel to G_1G_2 and at a distance y_1 from it. This line is now the neutral axis, since the moment of **T** with respect to it is zero. The moment of the couple is

$$C = \frac{MI}{R},$$

just as before. The bending moment is independent of the existence of an additional tension or thrust, but the position of the neutral axis is not.

221. Beams Subject to Perpendicular Forces.—Let a beam be held fixed at one end and be acted upon by forces **A** and **B** which are perpendicular to it, the weight of the beam being negligible. The beam is in equilibrium and is rigid enough to be essentially straight.

Consider any cross-section P on the left of **A** (Fig. 117). The part of the beam on the right of this section is in equilibrium. Call this section R and the rest of the beam L. The section of

the beam R is acted upon by **A, B,** and L. Hence, the action of L at P has a component **S** perpendicular to the beam, which is a shearing force and of such magnitude that

$$\mathbf{A} + \mathbf{B} + \mathbf{S} = 0,$$

and a couple, of which the moment is

$$C = aA - bB,$$

where a and b are the distances of **A** and **B** from P. Since

$$\mathbf{S} = -(\mathbf{A} + \mathbf{B}),$$

it is independent of the position of P so long as P is on the left of **A**. If the point P passes to the right of **A**, the shear **S** becomes equal to $-\mathbf{B}$ and remains constant until P passes **B** when it drops to zero.

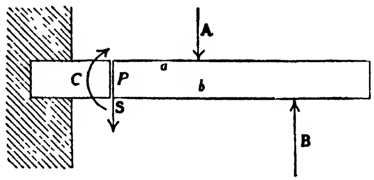


FIG. 117.

The magnitude of the couple changes as P moves to the right. If x is the distance that P moves to the right, then

$$\begin{aligned} C_x &= (a - x)A - (b - x)B = (aA - bB) - x(A - B) \\ &= C - x(A - B), \end{aligned}$$

and the couple is a linear function of x, provided

$$A \neq B.$$

When the point P passes to the right of **A**, the equation for the couple becomes

$$C_x = -(b - x)B.$$

The couple vanishes at **B** and remains zero to the end of the beam.

222. The Shearing Stresses and Bending Moments of a Loaded Beam.—Suppose a horizontal beam is supported at two points and supports weights W_1, W_2, $\cdots$. It is desired to find the shearing stress and bending moment at any point of the beam, neglecting the weight of the beam.

Let the beam be supported at the points A and B (Fig. 118) and let the weights be distributed as in the diagram. Let the distance AB be a, and the distances of the weights from A be x_1, x_2, and x_3. On taking moments about the points B and A, successively, it is found that

$$\begin{aligned} aW_A &= (a - x_1)W_1 + (a - x_2)W_2 + (a - x_3)W_3, \\ aW_B &= x_1W_1 + x_2W_2 + x_3W_3. \end{aligned}$$

Let P be any cross-section of the beam at a distance x from A; and suppose, at first, that it lies in the interval W_AW_1. The section of the beam AP is in equilibrium under the action of the support at A, $\mathbf{W}_A$, and the action of the part PB in and across

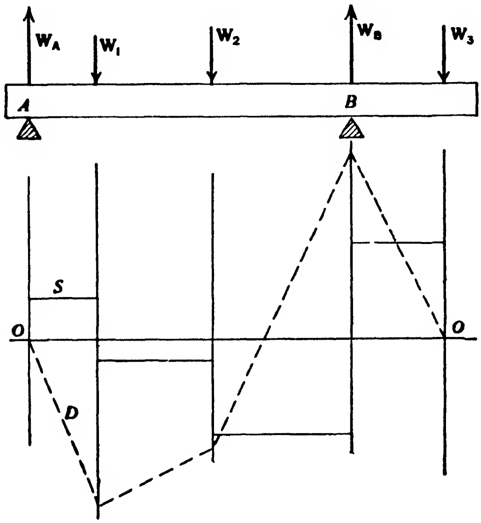


Fig. 118.

the plane P. This action at P can be resolved into a force $-\mathbf{W}_A$ and a couple, of which the moment is

$$C = xW_A.$$

Bearing in mind the conventions as to sign (Sec. 207), it is seen that the shearing stress is positive. The bending moment will be regarded as positive when the convex side of the beam is

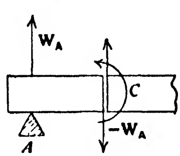


Fig. 119.

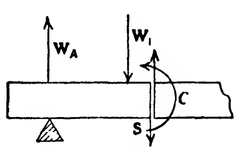


Fig. 120.

upward. Hence, the bending moment at P is negative. Let D be the bending moment and S the shearing stress. Then

$$-C = D = -xW_A, \qquad S = W_A.$$

It will be observed that S is constant, while D is a linear function of x.

Now let P be in the interval W_1W_2. Here then

$$S = W_A - W_1,$$
$$D = -xW_A + (x - x_1)W_1 = -x_1W_1 + x(-W_A + W_1).$$

The shear is again constant, and the bending moment is a linear function of x.

Likewise, in the interval W_2W_B,

$$S = W_A - W_1 - W_2,$$
$$D = -(x_1W_1 + x_2W_2) + x(-W_A + W_1 + W_2);$$

and, in the interval W_BW_3,

$$S = W_A + W_B - W_1 - W_2 = +W_3,$$
$$D = -(x_1W_1 + x_2W_2 - aW_B) + x(-W_A - W_B + W_1 + W_2),$$
$$= x_3W_3 - xW_3.$$

The shearing force and the bending moment are represented below the beam in Fig. 118 which has been drawn for the values

$$W_1 = 40, \quad W_2 = 50, \quad W_3 = 60, \quad W_A = 27, \quad W_B = 123,$$
$$x_1 = 4, \quad x_2 = 7, \quad x_3 = 12, \quad a = 10.$$

The bending moment is represented by the broken dotted line, while the shearing stress is represented by the discontinuous horizontal lines. It will be observed from the diagram that the left end of the beam is bent downward while the right end is bent upward. The point of inflection is at the point where the bending moment vanishes.

223. The Shearing Stresses and Bending Moments of Heavy, Uniform Beams.—In the previous section the weight of the beam was supposed to be negligible. In the present one, it will be assumed that the weight of the beam is the only force which is acting, aside from the supports of the beam.

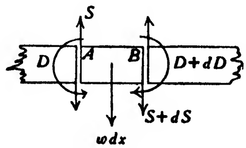


FIG. 121.

Let AB be a section of a horizontal beam of length dx and weight w per unit length, so that the force acting upon it, perpendicular to its length, is its weight wdx. Let S be the shearing stress and D the bending moment acting at A, and

$$S + dS \qquad \text{and} \qquad D + dD$$

the corresponding stresses acting at B.

Since the element AB is in equilibrium, a vertical resolution of the forces gives the equation

$$-(S + dS) + S - wdx = 0.$$

The equation of moments about the point B is

$$-(D + dD) + D - Sdx + \left(\frac{1}{2}dx\right)(wdx) = 0.$$

From these two equations, the following are derived:

$$\frac{dS}{dx} = -w, \qquad \frac{dD}{dx} = -S; \tag{1}$$

from which it is seen that the shearing stress and the bending moment of the element are defined by differential equations. The solution of these differential equations depends, naturally, upon the stresses existing in the ends of the beam, and these in turn depend upon the mode in which the beam is supported.

224. The Deflection of Heavy Horizontal Beams.—It was seen in Sec. 218 that the bending moment for a beam in the form of a circular arc is

$$D = \frac{MI}{R}.$$

Since this expression is independent of the length of the arc, it can be imagined that the length of the arc is indefinitely diminished, and, in fact, is the arc element ds of a curve. In this event, R becomes the radius of curvature ρ of the curve at the element ds. Hence, the bending moment of an elastic bar or beam at any point of the curve of equilibrium is

$$D = \pm\frac{MI}{\rho}.$$

If the x-axis is horizontal, the y-axis is vertical, and the radius of curvature positive when the curve is concave upward, it follows from the calculus that

$$\frac{1}{\rho} = \frac{\dfrac{d^2y}{dx^2}}{\left[1 + \left(\dfrac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}.$$

But since it is convenient to have the bending moment positive when the curve is concave downward, it follows that

$$D = -\frac{MI\dfrac{d^2y}{dx^2}}{\left[1 + \left(\dfrac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}.$$

If S is eliminated between the two equations of Eq. (223.1), it is found that

$$\frac{d^2D}{dx^2} = w,$$

and therefore

$$D = \frac{1}{2}wx^2 + B_1x + B_0,$$

where B_0 and B_1 are constants of integration. If these two expressions for D are equated, and if the constants k, c_2 and c_3 are defined by the relations

$$\frac{w}{MI} = k, \quad \frac{B_1}{MI} = -kc_3, \quad \frac{B_0}{MI} = -kc_2,$$

there results

$$\frac{\frac{d^2y}{dx^2}}{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}} = k\left(-\frac{1}{2}x^2 + c_3x + c_2\right).$$

This equation again is integrable, giving

$$\frac{\frac{dy}{dx}}{\sqrt{1 + \left(\frac{dy}{dx}\right)^2}} = k\left(-\frac{1}{6}x^3 + \frac{1}{2}c_3x^2 + c_2x + c_1\right) = k \cdot p(x)$$

which, on being solved for the derivative of y with respect to x, gives

$$\frac{dy}{dx} = \frac{kp}{\sqrt{1 - k^2p^2}} = kp + \frac{1}{2}k^3p^3 + \frac{1 \cdot 3}{2 \cdot 4}k^5p^5 + \cdots.$$

From the table of values of Young's modulus given in Sec. 208, it is seen that the value of the constant k is very small so that it is quite sufficient when the deflection is small to neglect the terms of higher degree and take

$$\frac{dy}{dx} = kp = k\left(-\frac{1}{6}x^3 + \frac{1}{2}c_3x^2 + c_2x + c_1\right);$$

therefore, $y = k\left(-\frac{1}{24}x^4 + \frac{1}{6}c_3x^3 + \frac{1}{2}c_2x^2 + c_1x + c_0\right),$

where $c_0, \cdots, c_3$ are the constants of integration which are determined by the conditions which the beam must satisfy.

225. Heavy Beam Supported at One End Only.—Let one end of the beam be held firmly both as to position and as to direction,

while the other end is entirely free. At the fixed and free ends of the beam,

$$x = 0 \qquad \text{and} \qquad x = l,$$

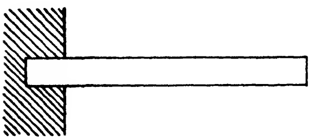


FIG. 122.

respectively. Also, at the fixed end,

$$y = 0 \qquad \text{and} \qquad \frac{dy}{dx} = 0.$$

At the free end, the shearing stress and the bending moment are both zero. Hence, the four constants of integration are determined by the four end conditions

$$\left.\begin{aligned} y &= 0 \\ \frac{dy}{dx} &= 0 \end{aligned}\right\} \text{at } x = 0 \qquad \text{and} \qquad \left.\begin{aligned} \frac{d^2y}{dx^2} &= 0 \\ \frac{d^3y}{dx^3} &= 0 \end{aligned}\right\} \text{at } x = l;$$

whence

$$\begin{aligned} c_0 &= 0, & -\frac{1}{2}l^2 + c_3 l + c_2 &= 0, \\ c_1 &= 0, & -l + c_3 &= 0. \end{aligned}$$

From the last column, it follows that

$$c_3 = l, \qquad c_2 = -\frac{1}{2}l^2;$$

and therefore the curve of the beam in equilibrium is

$$y = k\left(-\frac{1}{24}x^4 + \frac{1}{6}lx^3 - \frac{1}{4}l^2x^2\right).$$

The sag at the end of the beam is the value of y for x equal to l, or

$$\text{End sag} = -\frac{1}{8}kl^4.$$

226. Heavy Beam Supported at Both Ends.—If the beam is supported at both ends, so that the end conditions are

$$\left.\begin{aligned} y &= 0 \\ \frac{dy}{dx} &= 0 \end{aligned}\right\} \text{at } x = 0 \qquad \text{and} \qquad \left.\begin{aligned} y &= 0 \\ \frac{dy}{dx} &= 0 \end{aligned}\right\} \text{at } x = l,$$

then

$$\begin{aligned} c_0 &= 0, & -\frac{1}{24}l^4 + \frac{1}{6}c_3l^3 + \frac{1}{2}c_2l^2 &= 0, \\ c_1 &= 0, & -\frac{1}{6}l^3 + \frac{1}{2}c_3l^2 + c_2l &= 0. \end{aligned}$$

From the last two equations, are derived

$$c_3 = \frac{1}{2}l, \qquad c_2 = -\frac{1}{12}l^2.$$

The curve of the beam is therefore given by the equation

$$y = k\left(-\frac{1}{24}x^4 + \frac{1}{12}lx^3 - \frac{1}{24}l^2x^2 \right).$$

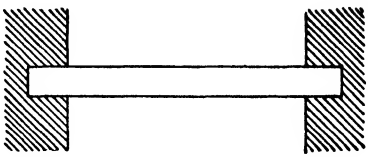


FIG. 123.

The value of y at the midpoint is the sag, namely,

$$\text{Sag} = -\frac{kl^4}{384}.$$

Aside from the constant factor $\pm MI$ (Sec. 222), the bending moment is the second derivative of y, and the shearing stress is the third derivative. Hence,

$$S = w\left(x - \frac{1}{2}l\right)$$

and

$$D = w\left(-\frac{1}{2}x^2 + \frac{1}{2}lx - \frac{1}{12}l^2 \right).$$

227. Elastic Wires with Applied Stresses at the Ends Only.—In the present section an investigation will be made of the curves which are assumed by naturally straight, elastic wires when a force and a couple in the same plane are applied at each end of the wire in such a manner, of course, that the wire is in equilibrium. A torsional stress is excluded, since the resulting curves would be in three dimensions and the subject would be too difficult for an elementary treatment.

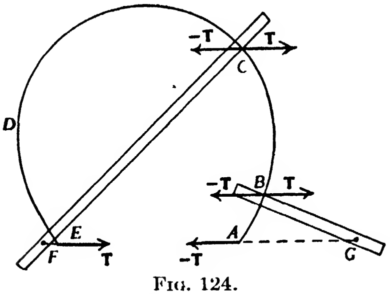


FIG. 124.

Let $ABCDE$ be an elastic rod, naturally straight, which has been bent and its two ends A and E connected by a tight string. The rod is in equilibrium under the action of the two external forces **T** and $-$**T**, which are acting on its ends only along the same string, and therefore in the same straight line. Let a rigid bar BG be rigidly attached to the wire at some point B, and let G be the point where the bar and the line of the string intersect. The triangle ABG can be regarded as a rigid triangle in equilibrium, or,

if there is any difficulty in imagining this, let the bar be replaced by a rigid triangle rigidly attached at B and A.

This rigid triangle is in equilibrium under the action of two forces, the tension of the string acting at A, and a stress acting at B due to the elastic rod CB. The point of application of the tension $\mathbf{T}$ acting at A can be moved to G without altering anything else, since, by hypothesis, the triangle is rigid. The triangle can now be cut down to the bar, as drawn in the diagram. Hence, the equilibrium of the rod $BCDE$ will not be disturbed if a rigid bar is rigidly attached to the rod at B, the string attached at the point G, and then the arc of the rod AB removed. A second rigid bar can be rigidly attached at, say, C and the string attached to the point F where the bar crosses the line of the string, so that the string now connects the points F and G. The rod CDE can be removed leaving the equilibrium of the rod BC still undisturbed.

The nature of the stresses at B and C can now be examined. Let y_B and y_C be the distances of B and C from the line FG. The tension $-\mathbf{T}$ acting at G is equivalent to a force $-\mathbf{T}$ acting at B and a couple of which the moment is $-y_BT$. Similarly, $+\mathbf{T}$ acting at F is equivalent to a force $+\mathbf{T}$ acting at C and a couple of which the moment is $+y_CT$.

Let x and y be the coordinates of any point of the arc BC, the x-axis being the line FG. It is evident that the bending moment at the point x, y, which is simply the value of the couple at that point, is

$$D = T \cdot y,$$

where T is a constant all along the arc, and is, in fact, the magnitude of the single force in the string which connects two points of the two bars.

If ρ is the radius of curvature at the point x, y, then (Sec. 224)

$$Ty = \frac{MI}{\rho}, \qquad \text{or} \qquad y = \frac{MI}{T} \cdot \frac{1}{\rho}.$$

It will improve the notation if the substitution

$$a^2 = \frac{4MI}{T}$$

is made, so that

$$y = \frac{a^2}{4\rho}.$$

Then

$$y = -\frac{\frac{a^2}{4}\frac{d^2y}{dx^2}}{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}$$

is the differential equation which is satisfied by the curve.

On multiplying through by $2dy/dx$ and then integrating, there results

$$y^2 = C + \frac{\frac{a^2}{2}}{\sqrt{1 + \left(\frac{dy}{dx}\right)^2}},$$

where C is the constant of integration.

Let θ be the angle which the tangent to the curve makes with the x axis. Then

$$\frac{dx}{ds} = \cos\theta, \qquad \frac{dy}{ds} = \sin\theta, \qquad \sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sec\theta;$$

and the above equation can be written

$$y^2 = C + \frac{1}{2}a^2 - a^2\sin^2\frac{1}{2}\theta = h^2 - a^2\sin^2\frac{1}{2}\theta,$$

where

$$h^2 = C + \frac{1}{2}a^2,$$

and h is evidently the maximum value of y^2.

Further integration depends upon the relative values of h and a, namely,

$$h^2 < a^2, \qquad h^2 = a^2, \qquad h^2 > a^2.$$

228. The Equation of the Elastic Curve for $h^2 < a^2$.—In the event that

$$h^2 < a^2,$$

let

$$h^2 = k^2a^2 \qquad (k^2 < 1); \qquad -\sin\frac{1}{2}\theta = k\sin\varphi;$$

then

$$y = h\cos\varphi.$$

In Fig. 125, let ABC be an arc of the curve and AC the line of tensions. With the midpoint O of AC as a center and h as a

radius, draw a semicircle which touches the curve at B. Let p be any point on the curve, and QpP be parallel to AC. Then

$$\varphi = \angle BOP,$$

since

$$y = h \cos \varphi;$$

also

$$\sin \theta = \frac{dy}{ds} = \frac{dy}{d\varphi} \cdot \frac{d\varphi}{ds} = -h \sin \varphi \frac{d\varphi}{ds},$$

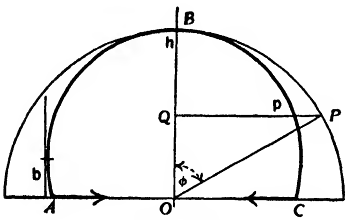


FIG. 125.

and therefore

$$\frac{ds}{d\varphi} = -h \frac{\sin \varphi}{\sin \theta}.$$

From the relation

$$-\sin \frac{1}{2}\theta = k \sin \varphi,$$

it is found that

$$-\sin \theta = 2k \sin \varphi \sqrt{1 - k^2 \sin^2 \varphi}.$$

Hence,

$$\frac{ds}{d\varphi} = \frac{a}{2} \cdot \frac{1}{\sqrt{1 - k^2 \sin^2 \varphi}},$$

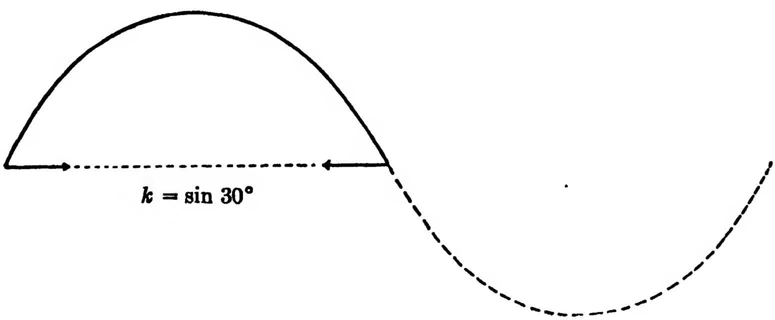


FIG. 126.

and

$$s = \frac{a}{2}\int_0^{\varphi} \frac{d\varphi}{\sqrt{1 - k^2 \sin^2 \varphi}} = \frac{1}{2} aF(k,\varphi),$$

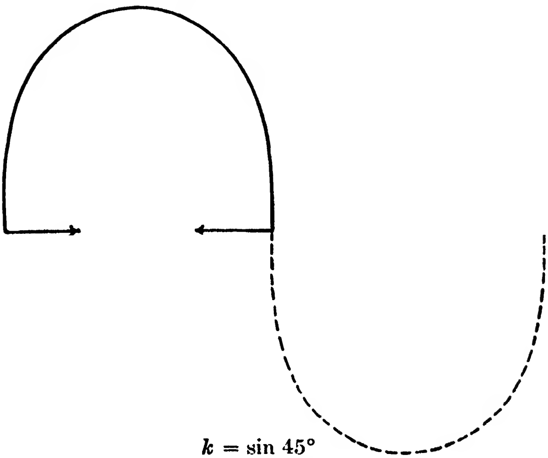
which is Legendre's elliptic integral of the first kind.

Since

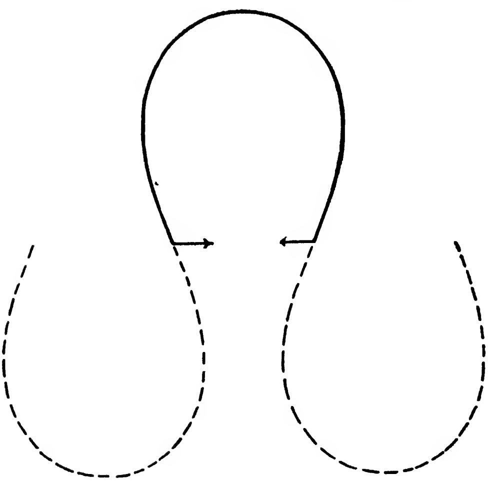
$$\cos \theta = \frac{dx}{ds},$$

it follows that

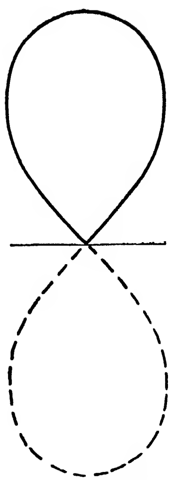
$$\frac{dx}{d\varphi} = \frac{ds}{d\varphi}\cos\theta = -h\frac{\sin\varphi}{\tan\theta},$$
$$= \frac{1}{2}a\frac{1 - 2k^2\sin^2\varphi}{\sqrt{1 - k^2\sin^2\varphi}};$$



$k = \sin 45°$
FIG. 127.



$k = \sin 55°$
FIG. 128.



$k = \sin 65° 22'$
FIG. 129.

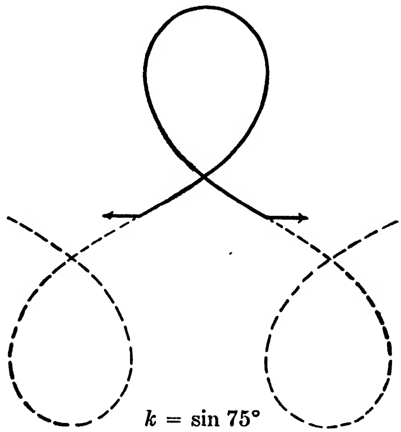
therefore

$$x = a\int_0^{\varphi}\sqrt{1 - k^2\sin^2\varphi}\,d\varphi - \frac{1}{2}a\int_0^{\varphi}\frac{d\varphi}{\sqrt{1 - k^2\sin^2\varphi}},$$
$$= a\cdot E(k, \varphi) - \frac{1}{2}a\cdot F(k, \varphi),$$

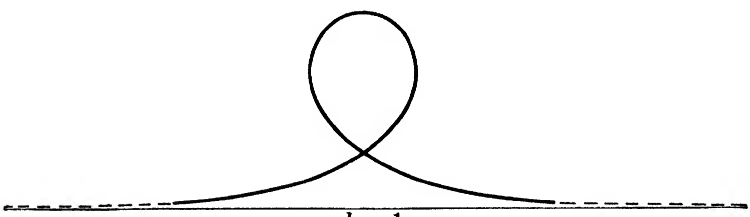
where $E(k, \varphi)$ is Legendre's elliptic integral of the second kind. On combining this expression for x with the equation

$$y = h \cos \varphi,$$

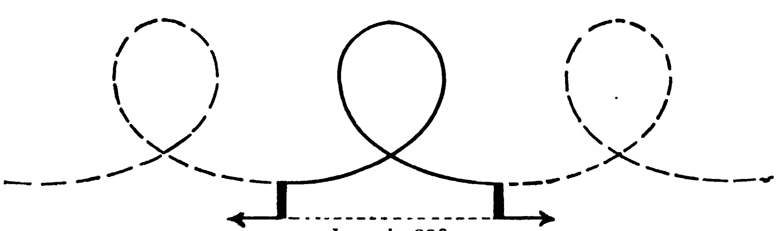
the parametric equations of the curve are obtained (see Figs. 126 to 130).



$k = \sin 75°$
FIG. 130.



$k = 1$
FIG. 131.



$k = \sin 80°$
FIG. 132.

Since y vanishes for φ equal to $\pi/2$, and since the second derivative vanishes with y, the curve has a point of inflection wherever it crosses the x-axis, or the line of tensions.

229. The Equation of the Elastic Curve for $h^2 = a^2$.—In case the constants h and a are equal the parameter k is equal to unity and the integrals for s and x are expressible in terms of the elementary functions. Thus,

$$y = a \cos \varphi,$$

$$x = a\int_0^{\varphi}\left[\cos \varphi - \frac{1}{2}\sec \varphi\right]d\varphi = a \sin \varphi - \frac{a}{2}\log \tan\left(\frac{\varphi}{2} + \frac{\pi}{4}\right),$$

and

$$s = \frac{a}{2}\int_0^{\varphi}\frac{d\varphi}{\cos \varphi} = \frac{a}{2}\log \tan\left(\frac{\varphi}{2} + \frac{\pi}{4}\right).$$

Since x tends toward infinity as φ approaches $\pi/2$, the complete curve has but a single loop the two branches of which are asymptotic to the x-axis (see Fig. 131).

230. The Equation of the Elastic Curve for $h^2 > a^2$.—If in Sec. 227, the substitution

$$a^2 = k^2h^2 \qquad (k^2 < 1),$$

is made, it is found that

$$y = h\sqrt{1 - k^2 \sin^2 \frac{1}{2}\theta},$$

from which it is evident that y never vanishes. Since the curvature vanishes only with y, it is clear that there are no points of inflection on the curve. The radius of curvature always is finite and of the same sign, and the tangent of the curve turns always in one direction.

On setting

$$\sin \varphi = -k \sin \frac{1}{2}\theta$$

and carrying through an analysis similar to that in Sec. 228, it is found that

$$y = h\sqrt{1 - k^2 \sin^2 \frac{1}{2}\theta}, \qquad s = \frac{1}{2}hk^2F\left(k, \frac{1}{2}\theta\right),$$

and

$$x = hE\left(k, \frac{1}{2}\theta\right) - h\left(1 - \frac{1}{2}k^2\right)F\left(k, \frac{1}{2}\theta\right).$$

(See Fig. 132.)

231. The Potential Energy of a Bent Elastic Rod.—The work done in bending a naturally straight elastic rod of length l to the arc of a circle of radius R is (Sec. 219)

$$W = \frac{MIl}{2R^2}.$$

If I is an element of arc ds and R is the radius of curvature, ρ, the work done in bending an arc element from its natural straightness to a curvature of $1/\rho$ is

$$dW = \frac{MI}{2\rho^2}ds,$$

and the total work done in bending the rod is the integral of this expression.

In the case of rods which have been bent to the shape of an elastic curve (Sec. 227)

$$\frac{1}{\rho} = \frac{4y}{a^2},$$

so that

$$dW = 2T\frac{y^2}{a^2}ds.$$

In the first case, which was discussed in Sec. 228,

$$\frac{y^2}{a^2} = k^2 \cos^2 \varphi, \qquad ds = \frac{\frac{1}{2}a\,d\varphi}{\sqrt{1 - k^2 \sin^2 \varphi}};$$

so that

$$dW = Ta\left[\sqrt{1 - k^2 \sin^2 \varphi} - \frac{(1 - k^2)}{\sqrt{1 - k^2 \sin^2 \varphi}}\right]d\varphi,$$

and

$$W = Ta[E(k, \varphi) - (1 - k^2)F(k, \varphi)] = T[x + (2k^2 - 1)s].$$

In the second case (Sec. 229) in which k^2 is equal to unity,

$$dW = Ta \cos \varphi d\varphi,$$

and

$$W = Ta \sin \varphi = T(x + s).$$

In the third case (Sec. 230),

$$\frac{y^2}{a^2} = \frac{\cos^2 \varphi}{k^2}, \qquad \frac{ds}{d\theta} = -\frac{1}{4}\,\frac{ak}{\cos \varphi}.$$

Therefore,

$$dW = -T\frac{a}{k} \cos \varphi \, d\frac{\theta}{2} = -Th\sqrt{1 - k^2 \sin^2 \frac{1}{2}\theta}\, d\frac{\theta}{2}.$$

The negative sign arises from the fact that, according to the conventions, θ is zero at the point B (Fig. 125), where s also is zero, and decreases as s increases. Hence, the integral must be taken from zero to some negative value of θ. It is more conven-

ient simply to change the sign of θ and integrate over a positive interval of the same numerical value. Then

$$dW = Th\sqrt{1 - k^2 \sin^2 \frac{1}{2}\theta}\, d\frac{\theta}{2}$$

and

$$W = Th \cdot E\left(k, \frac{\theta}{2}\right) = T\left[x + \left(\frac{2}{k^2} - 1\right)s\right].$$

It should not be forgotten that when φ is variable and a and k are constants, arcs of the same curve but of different lengths are dealt with. If it is desired to consider the work done in bending the same rod to different curvatures, then s is constant and a, k, and φ are variables.

232. Equations of Equilibrium of a Plane Bent Elastic Rod.—The elastic curve was derived under the hypothesis that no forces were acting upon the rod except upon its ends. If forces act upon each element of the rod as well as upon the ends, such as gravity, for instance, the problem is much more complicated.

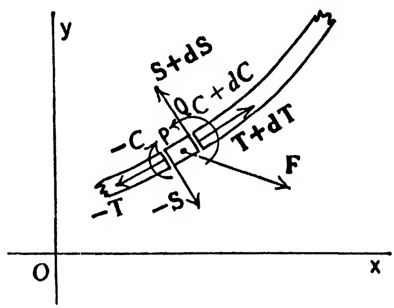


FIG. 133.

In Fig. 133, let T, S, and C be the magnitudes of the tension, shear, and couple of the part of the rod to the left upon the element ds, and $T + dT$, $S + dS$, and $C + dC$ be the corresponding actions of that part of the rod which joins ds on the right. Let Fds be the magnitude of the external force acting upon the element ds, so that F is the magnitude of the force per unit length of the rod, and $\mathbf{F}_t$ and $\mathbf{F}_n$ the components of $\mathbf{F}$ along the tangent and normal of the curve. Let the angle which the tangent makes with the x-axis be θ, so that

$$ds = \rho \cdot d\theta.$$

Then, on resolving the force along the tangent and along the normal and taking moments about the point P, the three following equations are obtained:

$$\begin{aligned}
-T + (T + dT)\cos d\theta - (S + dS)\sin d\theta + F_t ds &= 0,\\
-S + (S + dS)\cos d\theta + (T + dT)\sin d\theta + F_n ds &= 0,\\
-C + (C + dC) \qquad + \qquad (S + dS)ds &= 0.
\end{aligned}$$

On passing to the limit and replacing $d\theta$ by ds/ρ, there results

$$\frac{dT}{ds} - \frac{S}{\rho} + F_t = 0,$$
$$\frac{dS}{ds} + \frac{T}{\rho} + F_n = 0,$$
$$\frac{dC}{ds} \qquad + S = 0.$$

Since C is the bending moment,

$$C = \frac{MI}{\rho}.$$

If S is eliminated between these equations, it is found that

$$\frac{dT}{ds} + \frac{1}{\rho}\frac{dC}{ds} + F_t = 0$$

and

$$-\frac{d^2C}{ds^2} + \frac{1}{\rho}T + F_n = 0.$$

Let $\mathbf{F}_x$ and $\mathbf{F}_y$ be the components of $\mathbf{F}$ along the x- and y-axes and θ the angle which the tangent to the curve makes with the x-axis. Multiply the first of these two equations by $\cos\theta$, the second by $-\sin\theta$, and add. Then multiply the first equation by $\sin\theta$, the second by $\cos\theta$, and add. Then, since

$$\rho d\theta = ds,$$

it will be found that the resulting equations can be written

$$\frac{d}{ds}\left[\left(T - \frac{1}{\rho}C\right)\cos\theta\right] + \frac{d^2}{ds^2}(C\sin\theta) + F_x = 0,$$
$$\frac{d}{ds}\left[\left(T - \frac{1}{\rho}C\right)\sin\theta\right] - \frac{d^2}{ds^2}(C\cos\theta) + F_y = 0;$$

or, by integration,

$$\left(T - \frac{1}{\rho}C\right)\cos\theta + \frac{d}{ds}(C\sin\theta) = -\int F_x ds = -G_x,$$
$$\left(T - \frac{1}{\rho}C\right)\sin\theta - \frac{d}{ds}(C\cos\theta) = -\int F_y ds = -G_y.$$

On multiplying the first of these two equations by $\sin\theta$, the second by $-\cos\theta$ and adding, there is obtained

$$\frac{dC}{ds} = -G_x\sin\theta + G_y\cos\theta;$$

and since

$$\sin\theta = \frac{dy}{ds}, \qquad \cos\theta = \frac{dx}{ds},$$

it follows that

$$C = -\int G_x dy + \int G_y dx.$$

The equation

$$C = \pm\frac{MI}{\rho} = \pm\frac{MI\dfrac{d^2y}{dx^2}}{\left[\sqrt{1+\left(\dfrac{dy}{dx}\right)^2}\right]^3} = \pm MI\frac{d}{dx}\left(\frac{\dfrac{dy}{dx}}{\sqrt{1+\left(\dfrac{dy}{dx}\right)^2}}\right)$$

also holds. On equating these two values of C, there results finally as the differential equation of the curve

$$\pm MI\frac{d}{dx}\left(\frac{\dfrac{dy}{dx}}{\sqrt{1+\left(\dfrac{dy}{dx}\right)^2}}\right) = -\int G_x dy + \int G_y dx.$$

Problems XVIII

1. An overrigid framework in the shape of a square $ABCD$ and its diagonals is formed by six similar rods joined by smooth hinges. Two equal forces $\mathbf{T}$ act at A and B parallel to the sides AD and BC, and two opposite forces $-\mathbf{T}$ act at C and D. Find the change in the stress of each rod.

Ans.

$$\text{Tension in } AD = \frac{1}{2}\left(3 - \sqrt{2}\right) T;$$

$$\text{Tension in } AC = \frac{1}{2}\left(2 - \sqrt{2}\right) T;$$

$$\text{Thrust in } AB = \frac{1}{2}\left(\sqrt{2} - 1\right) T.$$

2. A rectangular gate $ABCD$ is formed by five similar rods smoothly jointed at the hinges. The sixth side AD is an upright post which allows the gate to swing. A small boy of weight w stands on the hinge at C, CD being the lower side of the gate. Find the changes in the stresses of the five rods due to the boy's weight. *Ans.* If a is the length of the horizontal rods, b the vertical end, d the diagonal, and if

$$f = \frac{a^3 + b^3 + d^3}{2a^3 + b^3 + 2d^3},$$

then the changes in the stresses are

$$+\frac{a}{b}(1-f)w \quad \text{in } AB, \qquad +\frac{d}{b}fw \quad \text{in } AC,$$

$$-\frac{d}{b}(1-f)w \quad \text{in } BD, \qquad -\frac{a}{b}fw \quad \text{in } CD,$$

and

$$+(1-f)w \quad \text{in } BC.$$

3. A steel beam projects 2 ft. beyond the wall which supports it. Its vertical depth at the wall is 12 in. and at its extremity 6 in. Its thickness is uniformly 3 in. It supports a weight of 50,000 lb. attached to its extremity. What is the deflection due to shear? *Ans.* 0.0038 in.

4. Prove that the droop of the ends of a plank which is supported at its center is only three-fifths of the droop which its center has when it is supported at its two ends.

5. Prove that if a plank is supported at its two ends the work done by gravity in bending the plank is only one-half of the work done in lowering its center of gravity.

6. A 10-lb. weight extends a helical spring 0.672 in. The spring consists of 20 coils, 2 in. in diameter, of wire 1/5 in. in diameter. What is the modulus of rigidity of the material used in the spring? *Ans.* 11.9×10^6.

7. A steel bar, 2 in. in diameter, for which Young's modulus is 29×10^6 lb. per square inch is bent into the arc of a circle of radius 400 ft. What is the maximum stress occurring at any point of a cross-section of the bar? *Ans.* 12,080 lb. per square inch.

8. If w is the weight of a beam per unit length, what is the maximum bending moment due to its own weight of a plank which is merely supported at its two ends? *Ans.* $wl^2/8$.

9. A beam is held firmly in a horizontal position at one end. The other end is supported in such a way that its sag is only one-half of what it would be if it were free. What part of the weight of the plank rests upon this support? *Ans.* 3/16.

10. The two ends of a very flexible rod of length l are connected by a tight string. Prove that the tension in the string is

$$T = 13.75 \cdot \frac{MI}{l^2},$$

if the tangents at the ends of the rod are parallel (Fig. 127); and

$$T = 21.56 \cdot \frac{MI}{l^2}$$

if the two ends are drawn into contact (Fig. 129).

11. Show that, if the two ends of a very flexible rod are in contact (Fig. 129), the angle between the two tangents is $81° 28'$.

PART III
THE DYNAMICS OF A PARTICLE

CHAPTER X

MOTION IN A STRAIGHT LINE

233. The Meaning of Dynamics.—The early students of mechanics recognized two types of force which were called *vis viva* and *vis mortua*, living force and dead force. The ideas lying behind these terms were somewhat vague, but in a general way the forces which were involved in equilibrium were dead forces while those which resulted in motion were living forces. Among modern students the term *vis mortua* has been discarded altogether, while the term *vis viva* has been preserved, but with a somewhat altered meaning. The distinction, between *vis mortua* and *vis viva* is still preserved, however, not in the nature of the forces themselves, but in the subjects of *statics* and *dynamics*. *Statics* is concerned with the study of forces which are in balance, in equilibrium, while *dynamics* is concerned with unbalanced forces, or forces which result in motion.

234. Uniform Motion in a Straight Line.—The simplest type of motion which can occur is uniform motion in a straight line. In accordance with Newton's first law of motion (Sec. 45) a body which is moving in this manner is not acted upon by any exterior force, or rather, more accurately, the resultant of all the forces which are acting upon it is zero. An elevator which is ascending with uniform speed is acted upon by gravity and friction in one direction and by the tension of the ropes in the opposite direction. The forces themselves are in equilibrium and their resultant is zero. It is just the same as though no forces were acting.

235. Non-uniform Motion.—If the motion is not uniform, even though it be a particle moving in a straight line, the resultant of the forces acting is not zero, and it is necessary to appeal to New-

ton's second law of motion for guidance. The motions of rigid and deformable bodies and of liquids and gases, which may be, and usually are, very complicated, can be resolved into the motions of their constituent particles. In order to analyze such motions, it is evident that a thorough knowledge of the dynamics of a single particle is necessary, and the remainder of the present volume will be devoted to this subject. The present chapter will consider the simplest type of such motion, namely, the motion of a particle in a straight line.

236. Velocity.—If a particle is in motion along a certain straight line, its position in that line at any instant t can be denoted by the letter s with the understanding that the coordinate s is measured from a conveniently chosen origin O in the line. Since the particle is in motion, s varies with the time; that is, s is a function of the time, and when it is desired to call attention to this fact, it will be written $s(t)$.

Its velocity at the instant t is (Sec. 28)

$$\lim_{\Delta t = 0} \frac{s(t + \Delta t) - s(t)}{\Delta t} = \frac{ds}{dt}. \tag{1}$$

Since the time is always the independent variable in mechanical problems, it is not necessary that the notation should always remind us of that fact, and it will be found that the notation

$$s' = \frac{ds}{dt}, \qquad s'' = \frac{d^2s}{dt^2},$$

which is similar to the fluxional notation of Newton, in which one or two dots were placed over the letter, is much simpler and more convenient. In this book, accents upon a letter will, with very few exceptions, indicate derivatives with respect to the time. In accordance with this notation the velocity of the particle in straight-line motion at any instant is the value of s' at that instant. For straight-line motion, *velocity* differs from *speed* only by the fact that speed is always positive, while velocity may be either positive or negative.

If the particle is moving toward the positive end of the straight line, it is seen from Eq. (1) that s' is positive, for $s(t + \Delta t)$ is *algebraically* greater than $s(t)$, whether the position of the particle s lies on the positive side of the origin or on the negative side. Therefore, wherever the particle may be in the straight line, a positive velocity can be represented by an arrow directed toward

the positive end of the coordinate axis, and a negative velocity by an arrow directed toward the negative end. The arrow represents the direction in which the particle is moving, and its speed.

237. Acceleration.—The acceleration at any instant is (Sec. 31)

$$\lim_{\Delta t = 0} \frac{s'(t + \Delta t) - s'(t)}{\Delta t} = s''.$$

The acceleration is positive if $s'(t + \Delta t)$ is algebraically greater than $s'(t)$. This is the case if the particle, wherever it may be in the line, is moving toward the positive end of the axis with increasing speed, or if it is moving toward the negative end with decreasing speed. Positive acceleration can be represented by an arrow directed toward the positive end of the axis. If the particle is moving toward the positive end of the axis with decreasing speed, or toward the negative end with increasing speed, the acceleration is negative and can be represented by an arrow directed toward the negative end of the axis.

For motion in a straight line, vector addition and subtraction are identical with algebraic addition and subtraction. It will not be forgotten, of course, that velocity and acceleration are vectors.

238. The Equation of Motion.—In accordance with Newton's second law of motion, the rate of change of momentum of the particle (that is, ms'', Sec. 43, where m is the mass of the particle) is proportional to the force which is acting upon it. If the units are properly chosen "proportionality" becomes equality, and therefore, for straight-line motion,

$$ms'' = \text{the force acting.}$$

In most of the problems which are taken up, the force which is acting depends upon the position of the particle, and is therefore a function of s, but it may also depend upon its velocity s' as when friction occurs, or it may depend upon the time itself, as in the case of sympathetic vibrations. Various constants also may be in evidence, but these can be neglected and the equation becomes:

$$ms'' = f(s, s', t). \tag{1}$$

This is called the *equation of motion*. From a physical point of view, it is the *force equation;* but from a mathematical point of view, it is a differential equation of the second order, and its

solution requires two constants of integration. On the physical side, these two constants of integration correspond to the fact that the law which expresses the force does not depend upon the particular position nor the particular velocity which the particle may have had at the particular moment $t = 0$. These particular values are called the *initial conditions*. The constants of integration are determined when the initial conditions are given; or, perhaps better, the constants of integration are determined by the initial conditions.

I. GRAVITY AND GRAVITATION

239. Freely Falling Bodies.—Every particle near the surface of the earth is attracted toward the earth by a force which, according to the law of gravitation, is

$$f = -k^2\frac{mE}{r^2},$$

where k^2 is a factor of proportionality which depends upon the units employed, m is the mass of the particle, E is the mass of the earth, and r is the distance from the center of the earth to the particle. For particles that remain near the surface of the earth, that is, within a thousand feet or so, the variation in r^2 is relatively so small that f is essentially constant. It can be written, therefore

$$f = -mg,$$

where mg is what is called the weight of the particle, and g is the acceleration of gravity (Sec. 50). The minus sign indicates that the positive end of the vertical line is upward, or toward the zenith.

If this expression for f is substituted in Eq. (238.1), it becomes

$$ms'' = -mg.$$

The mass factor m can be divided out (which shows that the motion is independent of the mass) leaving the *acceleration equation*

$$s'' = -g. \tag{1}$$

This equation is easily integrated, the first integral being

$$s' = v_0 - gt, \tag{2}$$

where v_0 is the constant of integration and indicates the value of the velocity at the instant $t = 0$. A second integration gives

$$s = s_0 + v_0t - \frac{1}{2}gt^2, \tag{3}$$

where s_0 is the constant of integration and denotes the initial value of the coordinate s. The values of s_0 and v_0 may be either positive or negative.

If Eq. (1) is multiplied by $2s'$, it becomes

$$2s's'' = -2gs',$$

each member of which is an exact differential; and the integral is

$$s'^2 = -2gs + c.$$

On imposing the initial conditions, namely,

$$\text{at } t = 0, \qquad s = s_0, \qquad s' = v_0,$$

it is found that

$$c = v_0^2 + 2gs_0,$$

and therefore

$$s'^2 - v_0^2 = -2g(s - s_0) \tag{4}$$

which is an *integral* of Eq. (1). This same equation, (4), can be obtained also by eliminating t between Eqs. (2) and (3). If the initial velocity v_0 is zero, and v denotes the velocity after the particle has fallen through the distance h, then

$$s' = v, \qquad s - s_0, = -h,$$

and Eq. (4) takes the form

$$v^2 = +2gh, \tag{5}$$

which is very useful and easy to remember.

If a particle is thrown straight upward, it will rise to a certain height, at which the velocity vanishes, and then fall. Thus the particle passes through the same points twice, once ascending and once descending. Equation (4) shows that for a given value of s, the value of s'^2 is uniquely determined. The speed, therefore, at a given point is the same for the descent as for the ascent.

The time at which the particle arrives at a given point is obtained from the solution of Eq. (3) for t. Since Eq. (3) is a quadratic equation in t, there are two solutions one of which gives the time at which the particle passes the given point in its upward motion, and the other in its downward motion.

240. The Resistance is Proportional to the Speed.—Suppose a particle is moving in a medium which opposes the motion with a resistance which is proportional to the speed of the particle, and that there is no other force acting. Then the acceleration equation is

$$s'' = -ks', \tag{1}$$

the minus sign indicating that the acceleration has a direction which is always opposite to the velocity s'. If Eq. (1) is integrated once, there results

$$s' = -ks + c_1.$$

If the particle is started at the origin with the speed v_0, then

$$c_1 = v_0,$$

and

$$s' = v_0 - ks. \tag{2}$$

On substituting

$$\sigma = \frac{v_0}{k} - s, \qquad \text{and therefore} \qquad \sigma' = -s',$$

Eq. (2) becomes

$$\sigma' = -k\sigma,$$

the solution of which is

$$\sigma = c_2 e^{-kt}.$$

Therefore,

$$s = \frac{v_0}{k} - c_2 e^{-kt}.$$

Since, by the initial conditions, s vanishes with t,

$$c_2 = \frac{v_0}{k},$$

and the solution, therefore, is

$$\left.\begin{aligned} s &= \frac{v_0}{k}(1 - e^{-kt}), \\ s' &= v_0 e^{-kt}. \end{aligned}\right\} \tag{3}$$

As the time increases the speed decreases and has the limit zero. The particle approaches asymptotically the point

$$s = \frac{v_0}{k}.$$

241. Falling Body with Resistance Proportional to the Speed. The acceleration equation in this case differs from Eq. (240.1) only by the addition of the term $-g$ to the right member, so that

$$s'' = -g - ks',$$

or

$$s'' + ks' + g = 0. \tag{1}$$

It will be assumed that the particle falls from rest from the height s_0, so that

$$\text{at } t = 0, \qquad s = s_0, \qquad s' = 0.$$

If the substitution

$$s' = \sigma - \frac{g}{k} \tag{2}$$

is made, where σ is the new dependent variable, Eq. (1) becomes

$$\sigma' + k\sigma = 0,$$

the solution of which is

$$\sigma = c_1 e^{-kt},$$

where c_1 is the constant of integration. Therefore, Eq. (2) becomes

$$s' = c_1 e^{-kt} - \frac{g}{k}.$$

Since s' vanishes when t vanishes

$$c_1 = \frac{g}{k},$$

and

$$s' = \frac{g}{k}e^{-kt} - \frac{g}{k}. \tag{3}$$

The solution is, therefore

$$s = s_0 + \frac{g}{k^2}(1 - e^{-kt}) - \frac{g}{k}t. \tag{4}$$

As the time increases the velocity approaches the limiting constant velocity $-g/k$, and the motion becomes approximately uniform motion in a straight line. The reason for this is that the resistance increases as the speed increases, while the acceleration due to gravity is constant. As the two accelerations are oppositely directed, their sum tends toward zero, and a zero acceleration is the condition for uniform motion.

If a raindrop should fall 4900 feet without the resistance of the air its speed at the ground would be 560 feet per second, and it would be highly dangerous to be caught out in a summer shower. A hailstorm would be more destructive than machine-gun fire. Owing to the resistance of the air the speed of a raindrop does not exceed 25 feet per second.

242. The Resistance is Proportional to the Square of the Speed.—For high speeds, the resistance of the air is more nearly proportional to the square of the speed than to the first power. Assuming this law of resistance, the acceleration equations are

$$\left.\begin{array}{lll} (a) & s'' = -g + k^2 g s'^2 & \text{(if the particle is falling),} \\ (b) & s'' = -g - k^2 g s'^2 & \text{(if the particle is rising).} \end{array}\right\} \quad (1)$$

Since the resistance changes sign with the velocity while s'^2 does not change sign with the velocity, the differential equation for the rising particle is different from that for the falling one. The equation for the falling particle will be integrated, assuming that it falls from rest from the height s_0.

On setting

$$\sigma = s', \tag{2}$$

the first equation (*a*) of Eq. (1) can be written

$$\frac{k\sigma'}{1 - k^2\sigma^2} = -kg, \tag{3}$$

the integral of which is

$$\tanh^{-1} k\sigma = -kgt, \tag{4}$$

the constant of integration being zero, since σ vanishes with t by virtue of the initial conditions.

On taking the hyperbolic tangent of both sides of Eq. (4) and then replacing σ by s' from Eq. (2), it is found that

$$s' = -\frac{1}{k} \tanh kgt. \tag{5}$$

The integral of this equation is

$$s = s_0 - \frac{1}{k^2 g} \log \cosh kgt. \tag{6}$$

Since $\tanh x$ approaches unity as x increases indefinitely, and

$$\log \cosh x = \log \frac{1}{2}(e^x + e^{-x}) = x + \log (1 + e^{-2x}) + \log \frac{1}{2},$$

it is seen from Eq. (5) that the speed of the particle increases from zero and has the limit $1/k$ which is constant; and from Eq. (6), that the motion becomes approximately uniform motion; but the limiting speed has, in general, a different value from what it would have had if the resistance had been proportional to the first power of the speed.

On multiplying through Eq. (3) by $2k\sigma$, it becomes

$$\frac{2k^2\sigma\sigma'}{1 - k^2\sigma^2} = -2k^2 g s',$$

both members of which are integrable. The integral is

$$\log (1 - k^2\sigma^2) = -2k^2 g(s_0 - s),$$

which, solved for σ^2, gives

$$s'^2 = \frac{1}{k^2}(1 - e^{-2k^2g(s_0 - s)}), \tag{7}$$

a relation which gives the speed in terms of the distance through which the particle has fallen. This equation holds only if

$$s < s_0.$$

243. Sliding Down an Inclined Plane.—If a particle is on a smooth inclined plane, whose angle of inclination to the plane of the horizon is α (Fig. 134), the component of its weight along the plane is $mg \sin \alpha$. The component $mg \cos \alpha$ perpendicular to the plane is balanced by the reaction of the plane. If the

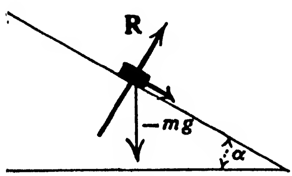


FIG. 134.

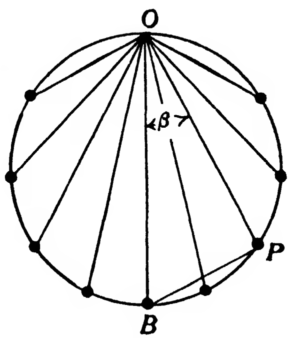


FIG. 135.

coordinate axis is taken along the line of greatest slope in the inclined plane, with the positive direction up the plane, the equation of acceleration is

$$s'' = -g \sin \alpha. \tag{1}$$

This equation differs from Eq. (239.1) only in that g of that equation is replaced in Eq. (1) by $g \sin \alpha$. It follows immediately, from the results of Sec. 239, that

$$s' = v_0 - g \sin \alpha \cdot t,$$

$$s = s_0 + v_0 t - \frac{1}{2} g \sin \alpha \cdot t^2,$$

and

$$s'^2 = v_0^2 + 2g \sin \alpha(s_0 - s).$$

Imagine a number of particles released at rest from the same point O at the same instant and that each particle slides down a plane whose inclination differs from that of all of the others. Let β be the angle which any one of the planes makes with the vertical, so that

$$\beta = 90 - \alpha.$$

Let the particle which falls vertically arrive at B at the instant t. The length OB is equal to $gt^2/2$. The particle which makes an angle β with the vertical, or α with the horizontal, arrives at the point P at the instant t, and the length OP is

$$OP = \frac{1}{2}gt^2 \sin \alpha = \frac{1}{2}gt^2 \cos \beta = OB \cos \beta.$$

The angle OPB, therefore, is a right angle, and P lies on the circle which has OB as a diameter. Since this is true, whatever the angle β may be, all of the particles lie on the same circle. It is true, also, whatever value t may have. Hence, *the locus of the particles at any instant is a circle the vertical diameter of which passes through the point O.* The diameter of this circle is proportional to the square of the time.

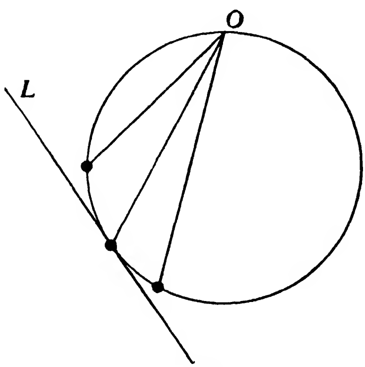


FIG. 136

This interesting fact enables us to answer easily the question: Along what straight line must a particle slide, starting from rest at O, in order that it may arrive at a given line L, whether straight or curved, in the least possible time? It is necessary only to construct the circle through O which is tangent to the given line and whose vertical diameter also passes through O; in other words, the vertical circle for which O is the highest point. The straight line joining O and the point of tangency is the line desired. If the line L is curved, there may be more than one such circle. In this event the one with the smallest diameter is to be chosen.

244. Atwood's Machine.—Let two weights w_1 and w_2 be tied to the two ends of a light string, and the string then thrown over a pulley which will be assumed to be without mass and frictionless. Since the string is inextensible, the motion of either of the weights is equal and opposite to the motion of the other.

For definiteness, let

$$w_2 > w_1.$$

Let s, measured from some convenient point, be the height of w_2 and let T be the tension in the string. Since

$$w_1 = m_1g, \qquad w_2 = m_2g,$$

the equations of motion are

$$-\frac{w_1}{g}s'' = -w_1 + T$$

$$+\frac{w_2}{g}s'' = -w_2 + T.$$

After multiplying the first equation by w_2, the second by w_1, and then adding, it is found that

$$T = \frac{2w_1w_2}{w_1 + w_2};$$

and if the first equation is subtracted from the second, it is found that

$$s'' = -\frac{w_2 - w_1}{w_2 + w_1}g.$$

It is easily verified that the tension is constant, and that

$$w_2 > T > w_1.$$

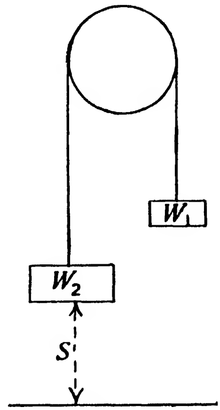


FIG. 137.

The acceleration s'' also is constant, and can be made as small as is desired by taking the two weights very nearly equal. Unfortunately, when the weights are very nearly equal and the forces are very nearly balanced, the frictional forces of the pulleys become very important instead of being negligible. If this were not so, the constant g could be determined very accurately by this method.

245. The Force Varies Inversely as the Square of the Distance. Let it be supposed that a particle is projected from the surface of the earth, and that it rises so high that the variations in the attraction of the earth cannot be neglected, as it was in the discussion of freely falling bodies (Sec. 239). Let it be assumed, also, that there is no atmosphere to disturb the motion.

If m is the mass of the particle, E the mass of the earth, s the distance of the particle from the center of the earth, then the equation of motion is

$$ms'' = -k^2\frac{mE}{s^2}.$$

If the factor m is removed and the substitution

$$k_1^2 = k^2E$$

is made, the acceleration equation is

$$s'' = -\frac{k_1^2}{s^2}. \tag{1}$$

At the surface of the earth

$$s'' = -g = -32.2 \qquad \text{and} \qquad s = 20{,}900{,}000 \text{ feet.}$$

Therefore,

$$k_1 = 1.186 \times 10^8, \qquad \log k_1 = 8.0745,$$

the units being one foot and one second.

On multiplying Eq. (1) by $2s'$ and integrating, there results

$$s'^2 = v^2 + 2k_1^2\left(\frac{1}{s} - \frac{1}{r}\right), \tag{2}$$

the constant of integration having been chosen so that

$$s' = v \qquad \text{when} \qquad s = r,$$

r being the radius of the earth. The particle continues to rise until its velocity s' vanishes, that is, until

$$\frac{1}{s} = \frac{1}{r} - \frac{v^2}{2k_1^2}. \tag{3}$$

If

$$v^2 < \frac{2k_1^2}{r}, \tag{4}$$

there exists a positive value of s for which Eq. (3) is satisfied, and for which, therefore, the velocity vanishes. In this event, the particle turns and falls back upon the earth.

If

$$v^2 = \frac{2k_1^2}{r}, \tag{5}$$

there does not exist a value of s for which the velocity vanishes, although as s increases the velocity s' diminishes and has zero as a limit. It is for this reason that the value of v which satisfies Eq. (5) is called the *velocity to infinity*.

If

$$v^2 > \frac{2k_1^2}{r}, \tag{6}$$

the value of s which satisfies Eq. (3) is negative, but as s is positive and increasing this value is never attained. The speed decreases as s increases, and has a limit which is positive and different from zero.

The further integration of Eq. (2) depends upon the value of the initial speed v.

246. Case I: $v^2 < 2k_1^2/r$.—Let the value of s for which the velocity vanishes be denoted by $2s_1$ so that, from Eq. (245.3)

$$\frac{1}{2s_1} = \frac{1}{r} - \frac{v^2}{2k_1^2}.$$

Then Eq. (245.2) becomes

$$s'^2 = 2k_1^2\left(\frac{1}{s} - \frac{1}{2s_1}\right) = k_1^2\left(\frac{2}{s} - \frac{1}{s_1}\right),$$

which solved for s' gives

$$s' = \pm\frac{k_1}{\sqrt{s_1}}\left(\frac{2s_1 - s}{s}\right)^{\frac{1}{2}}. \tag{1}$$

The plus sign is to be taken when the particle is ascending, and the minus sign when it is descending.

For the integration of Eq. (1) it is convenient to change the dependent variable from s to φ by the substitution

$$\left.\begin{aligned} s &= 2s_1 \sin^2 \frac{1}{2}\varphi, \\ s' &= 2s_1 \sin \frac{1}{2}\varphi \cos \frac{1}{2}\varphi \cdot \varphi'. \end{aligned}\right\} \tag{2}$$

Then Eq. (1) reduces to

$$2 \sin^2 \frac{1}{2}\varphi\, d\varphi = k_1 s_1^{-\frac{3}{2}} dt,$$

or

$$(1 - \cos \varphi)d\varphi = k_1 s_1^{-\frac{3}{2}} dt; \tag{3}$$

therefore,

$$\varphi - \sin \varphi = k_1 s_1^{-\frac{3}{2}} t, \tag{4}$$

if the constant of integration is chosen so that φ vanishes with t. The particle, of course, could not move below the surface of the earth, but there is no mathematical difficulty. Equation (3) shows that if the particle starts from the center of the earth, for which φ is zero, it does so with an infinite speed. At the surface of the earth the value of φ is given by Eq. (2)

$$r = 2s_1 \sin^2 \frac{1}{2}\varphi_r; \tag{5}$$

the time required to reach the surface from the center is, by Eq. (4),

$$T_r = \frac{s_1^{\frac{3}{2}}}{k_1}(\varphi_r - \sin \varphi_r). \tag{6}$$

At the highest point in its path the first equation of Eq. (2) shows that the value of ϕ is equal to π, and then Eq. (4) gives the time at which this point is attained, namely,

$$T_\pi = \frac{\pi s_1^{\frac{3}{2}}}{k_1}. \tag{7}$$

The time, therefore, from the surface of the earth to the highest point is

$$T = T_\pi - T_r.$$

247. Case II: $v^2 = 2k_1^2/r$.—If the initial velocity is the velocity to infinity, Eq. (245.2) has the very simple form

$$s'^2 = \frac{2k_1^2}{s}; \tag{1}$$

therefore,

$$\sqrt{s}ds = \sqrt{2}k_1dt;$$

so that

$$t = \frac{\sqrt{2}s^{\frac{3}{2}}}{3k_1}, \qquad s = \left(\frac{3k_1t}{\sqrt{2}}\right)^{\frac{2}{3}}, \tag{2}$$

if the constant of integration is chosen so that s vanishes with t.

248. Case III: $v^2 > 2k_1^2/r$.—If the initial velocity is greater than the velocity to infinity, then s_1 is negative (Eq. (245.3)), and Eq. (245.2) becomes

$$s'^2 = 2k_1^2\left(\frac{1}{s} + \frac{1}{2s_1}\right). \tag{1}$$

Therefore,

$$s' = \pm\frac{k_1}{\sqrt{s_1}}\left(\frac{2s_1 + s}{s}\right)^{\frac{1}{2}}. \tag{2}$$

For the integration of this equation, let

$$s = 2s_1 \sinh^2 \frac{1}{2}\varphi. \tag{3}$$

With this substitution, Eq. (2) passes over into

$$2 \sinh^2 \frac{1}{2}\varphi\, d\varphi = k_1s_1^{-\frac{3}{2}}dt,$$

or

$$(\cosh \varphi - 1)d\varphi = k_1s_1^{-\frac{3}{2}}dt,$$

the integral of which is

$$\sinh \varphi - \varphi = k_1s_1^{-\frac{3}{2}}t;$$

or

$$t = \frac{s_1^{\frac{3}{2}}}{k_1}(\sinh \varphi - \varphi), \tag{4}$$

the constant of integration having been chosen so that φ vanishes with t.

249. Projectiles to the Moon, and to Infinity.—If a projectile were shot from the surface of the earth with just sufficient speed to reach the moon, a distance of 240,000 miles, atmospheric resistance and other disturbances neglected, it would be found that

$$2s_1 = 240{,}000 \times 5280 = 1.267 \times 10^9 \text{ feet},$$

and therefore (Eq. (245.3))

$$v = 36{,}430 \text{ feet} = 6.90 \text{ miles per second}.$$

The time from the center of the earth to the moon is (Eq. (246.7))

$$T_\pi = 422{,}100 \text{ seconds} = 4 \text{ days } 21 \text{ hours } 15 \text{ minutes}.$$

In order to obtain the time from the surface of the earth, the time from the center of the earth to the surface must be subtracted from this result. From Eq. (246.5), it is found that, since the radius of the earth is 20,900,000 feet,

$$\varphi_r = 12° \; 20'.8;$$

and, from Eq. (246.6), it is found that

$$T_r = 228 \text{ seconds} = 3^\text{m} \; 48^\text{s}.$$

Hence, the time required for the projectile to reach the moon from the surface of the earth is

$$4^\text{d} \; 21^\text{h} \; 15^\text{m} - 3^\text{m} \; 48^\text{s} = 4^\text{d} \; 21^\text{h} \; 11^\text{m} \; 12^\text{s}.$$

Setting s equal to r in Eq. (247.1) and then solving for s', it is found that

$$s' = v = 36{,}730 \text{ feet} = 6.95 \text{ miles per second}.$$

This is called the *velocity to infinity*, or the *velocity of escape*. If the earth were the only body in the solar system, the projectile would recede indefinitely from the earth and its speed relative to the earth would eventually approach zero. It is interesting to notice that the velocity to the moon is only 300 feet per second less than the velocity to infinity.

250. Velocity of Escape and Surface Gravity of Other Planets. The equation

$$s'' = -\frac{k^2 E}{s^2},$$

Eq. (245.1), will hold for any other planet as well as for the earth, if the mass of the other planet is substituted for the mass of the earth; and likewise the integral

$$s'^2 = \frac{2k^2E}{s}, \tag{1}$$

which gives the velocity of escape. Let σ be the mean density of the earth and r its radius. At the surface of the earth, s'' is equal to $-g$, and therefore, assuming the earth to be a sphere

$$g = \frac{k^2E}{r^2} = \frac{4}{3}k^2\pi\sigma r, \tag{2}$$

since

$$E = \frac{4}{3}\pi\sigma r^3.$$

Likewise, if g_1, σ_1, and r_1 represent the corresponding quantities for any other planet,

$$g_1 = \frac{4}{3}k^2\pi\sigma_1 r_1. \tag{3}$$

The gravitational constant k^2 depends upon the units which are used, but its value is the same for all bodies. On taking the ratio of Eqs. (2) and (3), it is found that

$$g_1 = \frac{\sigma_1}{\sigma}\cdot\frac{r_1}{r}\cdot g,$$

which says that the surface gravity of any sphere or planet is proportional to the product of its radius and its mean density.

In the same manner, if v is the velocity of escape at the surface of the earth, Eq. (1) gives

$$v^2 = \frac{2k^2E}{r} = \frac{8}{3}k^2\pi\sigma r^2.$$

If v_1 is the velocity of escape on any other sphere or planet, then

$$v_1^2 = \frac{8}{3}k^2\pi\sigma_1 r_1^2.$$

The ratio of these two equations gives

$$v_1 = \sqrt{\frac{\sigma_1}{\sigma}}\cdot\frac{r_1}{r}\cdot v. \tag{4}$$

Since the radii of the various planets and their mean densities are known, their surface gravities and velocities of escape can be computed. The following table of their values is taken from Moulton's "Introduction to Astronomy:"

	Radius, miles	Mean density (water = 1)	Surface gravity (g = 1)	Velocity of escape, miles
Sun	432,200	1.40	27.64	384.3
Moon	1,080	3.34	0.16	1.49
Mercury	1,505	4.48?	0.31?	1.99
Venus	3,850	4.85?	0.85	6 51
Earth	3,959	5.53	1.00	6.95
Mars	2,170	3.58	0.36	3.22
Jupiter	44,200	1.25	2.52	38.04
Saturn	37,080	0.63	1.07	23.53
Uranus	15,100	1.44	0.99	14.40
Neptune	17,400	1.09	0.86	12.95

II. HARMONIC MOTION

251. Simple Harmonic Motion.—As a result of Hooke's law, one of the common forces that occurs in physics is a force that is proportional to the displacement of the particle from its position of equilibrium. Nearly all rapid vibrations are due to forces of this type of which the tuning fork and violin and piano strings are the most familiar examples.

Let the origin be taken at the position of equilibrium, and let the s-axis be in the line of the vibration. Since the force is proportional to the displacement, the equation of motion is

$$ms'' = -k^2 s. \tag{1}$$

The constant k^2 is the intensity of the force at a unit distance from the origin. For vibrating strings it depends upon the length, tension, and modulus of elasticity of the string. The force itself is always directed toward the origin, and since it is negative when s is positive, and positive when s is negative, Eq. (1) is valid on both sides of the origin.

In the present example, it will be observed that the mass factor does not divide out, since the force is independent of the mass; but the equation can be written

$$s'' = -\frac{k^2}{m}s = -k_1{}^2 s, \tag{2}$$

if

$$k_1 = \frac{k}{\sqrt{m}}.$$

The solution of Eq. (2) is simple, since in it the question is asked "What function of t reproduces itself, aside from a changed sign and a constant multiplier k_1^2, when it is differentiated twice?" The answer is immediate.

$$s = A \sin k_1 t, \qquad s = B \cos k_1 t,$$

or

$$s = A \sin k_1 t + B \cos k_1 t; \tag{3}$$

as this last form contains two independent constants of integration, it is the general solution. Other forms of the solution are

$$s = C \sin (k_1 t + c) \qquad s = D \cos (k_1 t + d); \tag{4}$$

and imaginary exponentials also are possible forms.

In all of these forms the solution is periodic with the period

$$P = \frac{2\pi}{k_1} = \frac{2\pi\sqrt{m}}{k}. \tag{5}$$

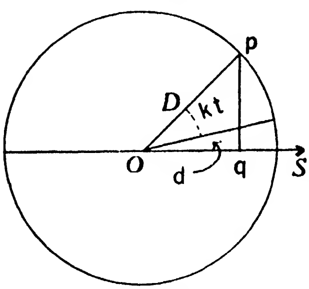


FIG. 138.

The amplitude of the oscillation, which is the maximum value of s, depends upon the constants of integration. In the form Eq. (4), the amplitude is C or D. It is to be noticed that the period is independent of the amplitude of the oscillation (which is very fortunate for violinists and pianists). For a given k, the period increases with the square root of the mass; and for a given mass it varies inversely with k.

If the solution is taken in the form

$$s = D \cos (k_1 t + d),$$

and a circle of radius D (Fig. 138) is drawn, for which the s-axis is a diameter with the origin at the center, and if a particle p describes this circle with uniform speed in the period,

$$P = \frac{2\pi}{k_1},$$

starting at the end of the radius which makes an angle d with the s-axis, then the position of the point q which is the projection of p on the s-axis is such that

$$Oq = D \cos (k_1 t + d) = s.$$

This simple geometrical illustration permits of the ready visualization of simple harmonic motion.

The velocity is

$$s' = -k_1 D \sin (k_1 t + d),$$

which is proportional to the line pq, the factor of proportionality being k_1. Hence,

$$s^2 + \frac{s'^2}{k_1^2} = D^2. \tag{6}$$

The integral Eq. (6) can also be obtained directly from Eq. (2) by multiplying through by $2s'$ and then integrating. If s and s' are regarded as the coordinates of a point, the integral Eq. (6) is the equation of an ellipse.

252. Tautochrone.—Given a field of force F and a curve C on which is fixed a point O. The curve C is said to be a *tautochrone* with respect to the point O, if a particle, starting from rest, constrained to move along C and moving under the action of the force F, arrives at the point O after an interval of time T which is independent of the point from which the particle starts.

In the case of simple harmonic motion, the length of time which is required for a particle which starts from rest to arrive at the origin is one-quarter of the period. That is,

$$T = \frac{\pi}{2k_1}$$

wherever the particle may have started. Since this expression does not depend upon the initial position, the motion is tautochronous.

Consider the converse proposition: For what laws of force is the straight line a tautochrone, assuming that the force depends upon the distance s only?

Let O be the point on the straight line for which the straight line is a tautochrone, and let $S(s)$ be the acceleration of the particle, the coordinate s being measured from the point O. Then the acceleration equation

$$s'' = S(s)$$

gives, on multiplying through by $2s'$ and integrating,

$$s'^2 = 2\int_{s_0}^{s} S ds.$$

Since S is a function of s alone the integral in the right member can be evaluated.

Let

$$\int_0^s S ds = -\varphi(s).$$

It is evident that the force must everywhere be directed toward the origin, and, therefore, S is negative for every positive value of s. Consequently, $\varphi(s)$ is a positive increasing function of s which vanishes with s. Then

$$s'^2 = 2(\varphi(s_0) - \varphi(s)),$$

and the time required for the particle to arrive at the origin is

$$T = \frac{1}{\sqrt{2}}\int_{s_0}^{0} \frac{ds}{\sqrt{\varphi(s_0) - \varphi(s)}}.$$

Let

$$\varphi(s) = \tau, \qquad \varphi(s_0) = \tau_0, \qquad s = \omega(\tau),$$

so that the function ω is the inverse of φ, and $\omega(\tau)$ vanishes with τ. Then

$$T = \frac{1}{\sqrt{2}}\int_{\tau_0}^{0} \frac{\frac{d\omega}{d\tau}d\tau}{\sqrt{\tau_0 - \tau}}.$$

In order that the straight line may be a tautochrone, it is necessary that T shall be independent of s_0, and therefore independent of τ_0. The condition necessary and sufficient that this may be true is that

$$\frac{\partial T}{\partial \tau_0} = 0.$$

In order that the limits of the integral shall not contain τ_0, set

$$\tau = \tau_0 u.$$

Then the integral becomes

$$T = \frac{1}{\sqrt{2}}\int_{1}^{0} \frac{\frac{d\omega}{d\tau}(\tau_0 u) \,.\sqrt{\tau_0}\,du}{\sqrt{1 - u}},$$

and its derivative with respect to τ_0 is

$$\begin{aligned}\frac{dT}{d\tau_0} &= \frac{1}{\sqrt{2}}\int_{1}^{0} \frac{\tau\frac{d^2\omega}{d\tau^2}(\tau_0 u) + \frac{1}{2}\frac{d\omega}{d\tau}(\tau_0 u)}{\sqrt{\tau_0}\sqrt{1 - u}}du \\ &= \frac{1}{\sqrt{2}}\int_{\tau_0}^{0} \frac{\tau\frac{d^2\omega}{d\tau^2} + \frac{1}{2}\frac{d\omega}{d\tau}}{\tau_0\sqrt{\tau_0 - \tau}}d\tau.\end{aligned}$$

Since this integral must be identically zero in τ_0, it is necessary that the integrand should be identically zero in τ; for if it were not identically zero in τ, one could take τ_0 so small that the integrand

would have the same sign between the limits 0 and τ_0, and then the integral would not be zero. Hence,

$$\tau \frac{d^2\omega}{d\tau^2} + \frac{1}{2}\frac{d\omega}{d\tau} = 0.$$

On setting

$$\frac{d\omega}{d\tau} = q,$$

this equation gives

$$\tau \frac{dq}{d\tau} + \frac{1}{2}q = 0;$$

whence

$$\sqrt{\tau} \cdot q = c_1, \qquad \omega(\tau) = 2c_1\sqrt{\tau} + c_2.$$

Since $\omega(\tau)$ vanishes with τ, the constant c_2 is zero; and, since $\omega(\tau)$ is s,

$$s = 2c_1\sqrt{\tau}, \qquad \tau = \frac{s^2}{4c_1^2},$$

which shows that the required function $\varphi(s)$ is

$$\varphi(s) = \frac{s^2}{4c_1^2}.$$

Since

$$S = -\frac{d\varphi}{ds} = -\frac{s}{2c_1^2},$$

the only law for which the straight line is a tautochrone is that in which the force is attractive and directly proportional to the distance.

253. Damped Harmonic Motion.—If the force which is acting upon the particle is attractive and directly proportional to the distance, and in addition there is a resistance which is proportional to the speed, the acceleration equation is

$$s'' = -k^2s - 2ls',$$

or

$$s'' + 2ls' + k^2s = 0. \tag{1}$$

This is a linear, homogeneous, differential equation with constant coefficients. The solution of such an equation can be reduced to the solution of an algebraic equation by the substitution

$$s = e^{\lambda t}.$$

By virtue of this substitution, Eq. (1) becomes

$$e^{\lambda t}(\lambda^2 + 2l\lambda + k^2) = 0,$$

and therefore

$$\lambda^2 + 2l\lambda + k^2 = 0.$$

This equation gives two values of λ, provided its discriminant $l^2 - k^2$ is not zero. These values are

$$\lambda_1 = -l + \sqrt{l^2 - k^2}; \qquad \lambda_2 = -l - \sqrt{l^2 - k^2}.$$

The general solution of Eq. (1), therefore, is

$$\begin{aligned} s &= Ae^{(-l+\sqrt{l^2-k^2})t} + Be^{(-l-\sqrt{l^2-k^2})t} \\ &= e^{-lt}[Ae^{+\sqrt{l^2-k^2}t} + Be^{-\sqrt{l^2-k^2}t}]. \end{aligned} \tag{2}$$

If the resistance is small, and therefore l is small, so that

$$l^2 - k^2 < 0,$$

it is convenient to set

$$\sqrt{l^2 - k^2} = in, \qquad \text{where} \qquad i = \sqrt{-1},$$

and then

$$s = e^{-lt}[Ae^{int} + Be^{-int}]. \tag{3}$$

Since

$$e^{int} = \cos nt + i \sin nt \qquad e^{-int} = \cos nt - i \sin nt,$$

Eq. (3) can also be written

$$s = e^{-lt}[(A + B) \cos nt + i\,(A - B) \sin nt];$$

or again by setting

$$A + B = C, \qquad i(A - B) = D,$$

it becomes

$$s = e^{-lt}[C \cos nt + D \sin nt]. \tag{4}$$

This expression differs from that of pure harmonic motion (Eq. (251.3)) only by the *damping factor* e^{-lt}. Since this factor diminishes as the time increases the amplitude of the oscillations diminishes, and the particle eventually comes to rest in its position of equilibrium; but in the meantime the period of the oscillations remains constant, namely,

$$P = \frac{2\pi}{n} = \frac{2\pi}{\sqrt{k^2 - l^2}}.$$

This period, it will be observed, is longer than it would be if there were no resistance, and the greater the resistance the longer the period, provided of course $k^2 - l^2$ is positive.

If

$$k^2 - l^2 = 0,$$

the two values of λ are equal, and

$$\lambda_1 = \lambda_2 = -l.$$

There is but one solution of the form

$$s = Ae^{-lt}.$$

It will be found, however, that, if l equals k, the critical value of l for which oscillation ceases,

$$s = e^{-lt}(At + B), \tag{5}$$

satisfies Eq. (1), and since it contains two constants of integration, it is the general solution. If A and B are determined so as to satisfy the initial conditions

$$s = s_0, \qquad s' = 0,$$

it is found that

$$s = s_0(1 + lt)e^{-lt}, \tag{6}$$

and the particle never passes through the origin.

If

$$l^2 - k^2 > 0,$$

both values of λ are real and negative; λ_1, however, tends toward zero as l increases. If the initial conditions are

$$s = s_0 \qquad s' = 0,$$

the solution is

$$s = \frac{s_0 e^{-lt}}{2\sqrt{l^2 - k^2}}[(l + \sqrt{l^2 - k^2})e^{\sqrt{l^2-k^2}\,t} + (-l + \sqrt{l^2 - k^2})e^{-\sqrt{l^2-k^2}\,t}]$$

and

$$s' = \frac{s_0 k^2 e^{-lt}}{2\sqrt{l^2 - k^2}}[e^{-\sqrt{l^2-k^2}\,t} - e^{+\sqrt{l^2-k^2}\,t}]. \tag{7}$$

For large values of l the velocity is negative and very slow.

III. CONSERVATIVE FORCES IN GENERAL

254. The General Case for Conservative Forces.—If the given force depends upon the position of the particle only, the equation of motion is

$$ms'' = f(s), \tag{1}$$

where $f(s)$, which will be assumed to be a continuous function of s, is the given force. On multiplying through by s' and then integrating, there results:

$$\left.\begin{aligned} \frac{1}{2}ms'^2 &= \int f(s)ds + \text{constant}, \\ &= -U(s) + E, \end{aligned}\right\} \tag{2}$$

which is called the energy equation, the left member $ms'^2/2$ being the kinetic energy, $U(s)$ the potential energy, and the constant E the total energy.

Since the kinetic energy is essentially positive, it is evident that s can have only such values as make the right member also positive. If s_0 satisfies this condition, and is the initial value of s, then

$$t = \pm \int_{s_0}^{s} \frac{ds}{\sqrt{\frac{2}{m}E - \frac{2}{m}U(s)}}. \tag{3}$$

For simplicity of notation, let

$$\frac{2}{m}[E - U(s)] = \varphi(s),$$

so that

$$s' = \pm\sqrt{\varphi(s)}. \tag{4}$$

Let the graph of $\varphi(s)$ be represented by Fig. 139, and let s_1, s_2, and s_3 be values of s for which $\varphi(s)$ vanishes. Then the initial value of s may lie between s_1 and s_2, or to the right of

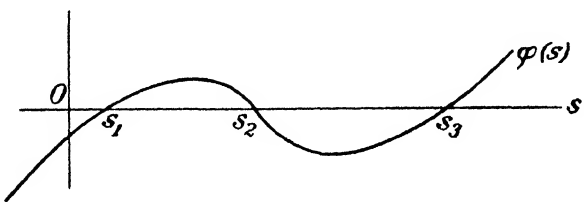


FIG. 139.

s_3 since $\varphi(s)$ is positive in these intervals, but it cannot lie between s_2 and s_3 for real values of s'. Suppose, for definiteness, that $\varphi(s)$ is a rational fraction and that s_0 lies between s_1 and s_2. If the particle is moving toward the right, s' is positive and the positive sign is to be taken before the radical in Eq. (4).

Since $\varphi(s)$ vanishes at s_2, it can be factored,

$$\varphi(s) = (s_2 - s)^n\psi(s),$$

where $\psi(s)$ is a rational fraction. If no root of the numerator or denominator of $\psi(s)$ lies between s_1 and s_2, $\psi(s)$ is continuous and positive between s_1 and s_2. Let A^2 and B^2 be the minimum and maximum values of $\psi(s)$ in this interval; then, provided $n = 1$, that is, s_2 is a simple root of $\varphi(s)$,

$$A\sqrt{s_2 - s} \leqq \sqrt{\varphi(s)} \leqq B\sqrt{s_2 - s}, \qquad \text{for} \qquad s_0 \leqq s \leqq s_2.$$

where α_x, α_y, α_z are the x-, y-, and z-components of the acceleration α.

In Fig. 141, let p_1 be the position of the particle at the instant t and $\mathbf{v}_1$ its velocity; let p_2 and $\mathbf{v}_2$ be the position and velocity of the particle at the instant $t + \Delta t$. If the vector $\mathbf{v}_2$ is drawn with p_1 as origin, it is seen that

$$\mathbf{v}_2 - \mathbf{v}_1 = p_1 q.$$

Then, if

$$\mathbf{a} = \frac{p_1 q}{\Delta t},$$

the vector $\mathbf{a}$ is the average acceleration in the interval Δt, and if α is the limit of $\mathbf{a}$ as Δt diminishes, then α is the acceleration of the particle at p_1.

The *osculating plane* of the curve at p_1 is defined to be the limiting position of the plane which passes through the tangent at p_1 and any neighboring point p_2, as p_2 approaches p_1. The *principal normal of the curve* lies in the osculating plane, and the *binormal* of the curve is perpendicular to the osculating plane. From these definitions, it is evident that both the velocity and acceleration vectors lie in the osculating plane.

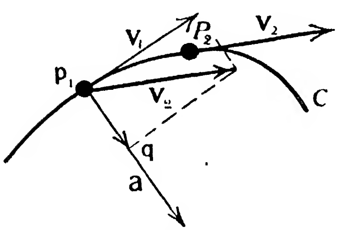


FIG. 141.

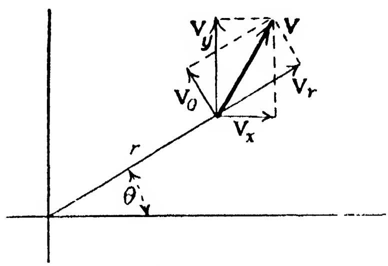


FIG. 142.

257. Polar Coordinates in the Plane.—Let $\mathbf{V}$ be any vector in the xy-plane (Fig. 142), and let r and θ be the polar coordinates of its origin. Let $\mathbf{V}_x$ and $\mathbf{V}_y$ be its components parallel to the x- and y-axes, and $\mathbf{V}_r$ and $\mathbf{V}_\theta$ its components along the radius vector to its origin and perpendicular to this radius vector. Let $\mathbf{V}_{xr}$ and $\mathbf{V}_{x\theta}$ be the components of $\mathbf{V}_x$, and $\mathbf{V}_{yr}$ and $\mathbf{V}_{y\theta}$ the components of $\mathbf{V}_y$ along and perpendicular to the radius vector. Then

$$\mathbf{V}_r = \mathbf{V}_{xr} + \mathbf{V}_{yr}$$
$$\mathbf{V}_\theta = \mathbf{V}_{x\theta} + \mathbf{V}_{y\theta}.$$

Since

$$V_{xr} = V_x \cos\theta, \qquad V_{x\theta} = -V_x \sin\theta,$$
$$V_{yr} = V_y \sin\theta, \qquad V_{y\theta} = +V_y \cos\theta,$$

it follows that

$$\left.\begin{aligned} V_r &= +V_x \cos\theta + V_y \sin\theta, \\ V_\theta &= -V_x \sin\theta + V_y \cos\theta. \end{aligned}\right\} \tag{1}$$

Now let the vector **V** be the velocity **v**. Then

$$v_x = x', \qquad v_y = y',$$

and, since

$$x = r\cos\theta, \qquad y = r\sin\theta,$$

it is found by differentiation that

$$\left.\begin{aligned} v_x &= x' = r'\cos\theta - r\theta'\sin\theta, \\ v_y &= y' = r'\sin\theta + r\theta'\cos\theta. \end{aligned}\right\} \tag{2}$$

On substituting Eq. (2) in Eq. (1), there results

$$v_r = r', \qquad v_\theta = r\theta', \tag{3}$$

from which it is learned that the component of the velocity along the radius vector is r', and the component perpendicular to the radius vector is $r\theta'$.

Next let the vector **V** be the acceleration $\boldsymbol{\alpha}$. Then

$$\begin{aligned} \alpha_x &= x'' = (r'' - r\theta'^2)\cos\theta - (r\theta'' + 2r'\theta')\sin\theta, \\ \alpha_y &= y'' = (r'' - r\theta'^2)\sin\theta + (r\theta'' + 2r'\theta')\cos\theta. \end{aligned} \tag{4}$$

On substituting Eq. (4) in Eq. (1), it is found that

$$\begin{aligned} \alpha_r &= r'' - r\theta'^2, \\ \alpha_\theta &= r\theta'' + 2r'\theta' = \frac{1}{r}(r^2\theta')' \end{aligned} \tag{5}$$

are the expressions for the components of the acceleration along the radius vector and perpendicular to it, in terms of the polar coordinates and their derivatives.

258. The Intrinsic Equations.—It is convenient at times to resolve the acceleration into components along the tangent to the path of the particle and along the normal.

Since the acceleration of the particle is the velocity of the terminus of the velocity vector in the hodograph, the components of $\boldsymbol{\alpha}$ in the direction of **v** (or, along the tangent to the path) and perpendicular to it (or, along the normal) are, by Eq. (257.3)

$$\alpha_t = v', \qquad \alpha_n = v\omega',$$

where v and ω are the polar coordinates of the hodograph. Since ω is the angle which the tangent to the path makes with the x-axis, and since

$$\rho\, d\omega = ds,$$

or,

$$\rho\omega' = s' = v,$$

where ρ is the radius of curvature of the path and ds is the arc element, it follows that

$$\omega' = \frac{v}{\rho},$$

and therefore the components of the acceleration along the tangent and along the normal are

$$\alpha_t = v' = s'', \qquad \alpha_n = \frac{v^2}{\rho}. \tag{1}$$

These are called the *intrinsic equations* of the curve, or motion, since they are independent of the coordinate system. It is evident that if the force is always perpendicular to the path of

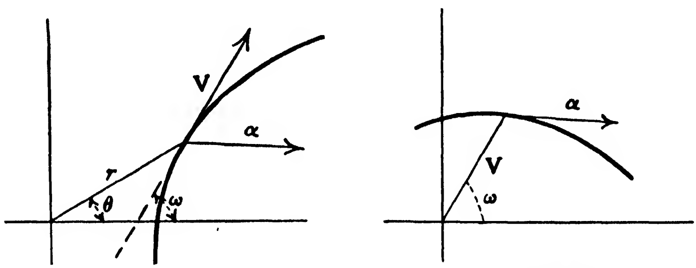


Fig. 143.

motion, the tangential component of the acceleration is always zero, and therefore the speed of the particle is constant. The acceleration vector, if it is not zero, is always directed towards the concave side of the curve. It also lies in the osculating plane (Sec. 256), if the path is not a plane curve, and therefore its component along the binormal is always zero.

259. Other Relations.—It will be of interest to project the tangential and normal components of the acceleration upon the x-, y-, and z-axes, and thus verify the expression already derived (Eq. (256.1)), for α_x, α_y, and α_z. It will be remembered that the acceleration along the binormal, which is perpendicular to the osculating plane, is zero.

Let α, β, and γ be the direction cosines of the tangent to the path, and λ, μ, and ν the direction cosines of the principal normal to the curve (that is, the normal in the osculating plane).

Then

$$\left.\begin{aligned} \alpha_x &= \alpha v' + \lambda\frac{v^2}{\rho}, \\ \alpha_y &= \beta v' + \mu\frac{v^2}{\rho}, \\ \alpha_z &= \gamma v' + \nu\frac{v^2}{\rho}. \end{aligned}\right\} \tag{1}$$

It will be assumed as well known that

$$\alpha = \frac{dx}{ds}, \qquad \beta = \frac{dy}{ds}, \qquad \gamma = \frac{dz}{ds}, \tag{2}$$

and it will be proved that

$$\lambda = \rho\frac{d^2x}{ds^2}, \qquad \mu = \rho\frac{d^2y}{ds^2}, \qquad \nu = \rho\frac{d^2z}{ds^2}. \tag{3}$$

Imagine a particle moving along the given path with a constant speed which is equal to unity. Since the speed is constant, the tangential component of the acceleration is zero (Eq. (258.1)) and therefore the acceleration has the same direction as the principal normal. Also, since the speed is constant, the hodograph is a curve on the surface of a sphere of radius unity.

Let $P(x, y, z)$ be a point on the path of the particle and $Q\ (\alpha, \beta, \gamma)$ be the corresponding point on the hodograph. Let P_1 be a point on the path near P, and Q_1 the corresponding point on the hodograph. Then

$$PP_1 = ds, \qquad QQ_1 = d\sigma.$$

The radius vector **v** to the point Q is parallel to the tangent at P; and the tangent to the hodograph $\boldsymbol{\alpha}$ at Q is parallel to the principal normal at P, and therefore has the same direction cosines. Hence, by Eq. (259.2),

$$\lambda = \frac{d\alpha}{d\sigma}, \qquad \mu = \frac{d\beta}{d\sigma}, \qquad \nu = \frac{d\gamma}{d\sigma};$$

or,

$$\left.\begin{aligned} \lambda &= \frac{d\alpha}{ds}\cdot\frac{ds}{d\sigma} = \frac{d^2x}{ds^2}\cdot\frac{ds}{d\sigma} = \rho\frac{d^2x}{ds^2}, \\ \mu &= \frac{d\beta}{ds}\cdot\frac{ds}{d\sigma} = \frac{d^2y}{ds^2}\cdot\frac{ds}{d\sigma} = \rho\frac{d^2y}{ds^2}, \\ \nu &= \frac{d\gamma}{ds}\cdot\frac{ds}{d\sigma} = \frac{d^2z}{ds^2}\cdot\frac{ds}{d\sigma} = \rho\frac{d^2z}{ds^2}. \end{aligned}\right\} \tag{4}$$

The equalities in the last column follow from the fact that

$$ds = \rho d\omega = PP_1, \qquad d\omega = d\sigma = QQ_1;$$

that is, the angle through which the tangent at P turns is the same as the angle through which the radius vector of the hodograph turns, the length of the radius vector being unity.

Returning now to Eq. (259.1), the expressions for the rectangular components of the acceleration become

$$\left.\begin{aligned} \alpha_x &= v'\frac{dx}{ds} + v^2\frac{d^2x}{ds^2} = \frac{d}{dt}\left(v\frac{dx}{ds}\right) = x'', \\ \alpha_y &= v'\frac{dy}{ds} + v^2\frac{d^2y}{ds^2} = \frac{d}{dt}\left(v\frac{dy}{ds}\right) = y'', \\ \alpha_z &= v'\frac{dz}{ds} + v^2\frac{d^2z}{ds^2} = \frac{d}{dt}\left(v\frac{dz}{ds}\right) = z'', \end{aligned}\right\} \tag{5}$$

just as before (Eq. 256.1).

260. Components of a Vector in Spherical Coordinates.—The polar coordinates of a point in three dimensions are

$$\left.\begin{aligned} x &= r\cos\varphi\cos\theta, \\ y &= r\cos\varphi\sin\theta, \\ z &= r\sin\varphi, \end{aligned}\right\} \tag{1}$$

where φ and θ are angles which correspond to latitude and longitude on a sphere. It is desired to find the components of velocity along the radius vector and in two perpendicular directions in terms of r, φ, and θ and their derivatives.

For this purpose a new rectangular set of axes ξ, η, and ζ will be defined, such that the ξ-axis coincides with the radius vector at the instant t. The ζ-axis lies in the plane which contains the z-axis and the ξ-axis; and the η-axis is perpendicular to the ξ- and ζ-axes. Like the xyz-trihedron, the $\xi\eta\zeta$-trihedron will be assumed to be right handed. The table of direction cosines will be the same as that used in Sec. 69, namely,

	ξ	η	ζ
x	α_1	α_2	α_3
y	β_1	β_2	β_3
z	γ_1	γ_2	γ_3

(2)

Let $\mathbf{V}$ be any vector, $\mathbf{V}_x$, $\mathbf{V}_y$, $\mathbf{V}_z$ its x-, y-, and z-components, and $\mathbf{V}_\xi$ $\mathbf{V}_\eta$ $\mathbf{V}_\zeta$ its ξ-, η-, and ζ-components. Then

$$\left.\begin{aligned} \mathbf{V}_\xi &= \alpha_1\mathbf{V}_x + \beta_1\mathbf{V}_y + \gamma_1\mathbf{V}_z, \\ \mathbf{V}_\eta &= \alpha_2\mathbf{V}_x + \beta_2\mathbf{V}_y + \gamma_2\mathbf{V}_z, \\ \mathbf{V}_\zeta &= \alpha_3\mathbf{V}_x + \beta_3\mathbf{V}_y + \gamma_3\mathbf{V}_z. \end{aligned}\right\} \tag{3}$$

If $\mathbf{V}$ is taken to be the velocity of the particle and $\mathbf{V}_x$, $\mathbf{V}_y$, and $\mathbf{V}_z$ its components along the x-, y-, and z-directions, $\mathbf{V}_\xi$ is the component of the velocity along the radius vector and $\mathbf{V}_\eta$ and $\mathbf{V}_\xi$ are the components in two perpendicular directions, so that

$$\mathbf{v}_r = \mathbf{V}_\xi, \qquad \mathbf{v}_\theta = \mathbf{V}_\eta, \qquad \mathbf{v}_\varphi = \mathbf{V}_\zeta.$$

In order that $\mathbf{v}_r$, $\mathbf{v}_\varphi$, and $\mathbf{v}_\theta$ may be expressed in terms of r, φ, and θ and their derivatives, it is necessary that $\mathbf{v}_x$, $\mathbf{v}_y$, $\mathbf{v}_z$ and the direction cosines $\alpha_1, \cdot \cdot \cdot, \gamma_3$ be expressed in the same way.

261. The Nine Direction Cosines as Functions of the Polar Angles.—In any spherical triangle of which the angles are A, B, and C and the sides opposite these angles are a, b, and c, the three following relations hold:

$$\left.\begin{aligned} \cos a &= \cos b \cos c + \sin b \sin c \cos A, \\ \cos B \sin a &= \cos b \sin c - \sin b \cos c \cos A, \\ \sin B \sin a &= \sin b \sin A. \end{aligned}\right\} \tag{1}$$

If any three parts of the triangle are given, the other three parts are determined by these equations which are valid for all spherical triangles.

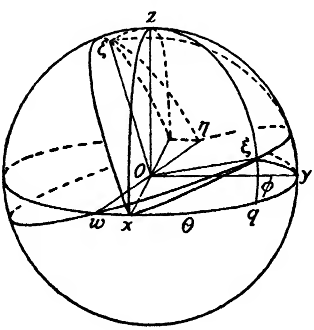


FIG. 144.

In Fig. 144, the x-, y-, and z-coordinates are drawn in their usual position. About O as a center is drawn a sphere of radius unity. Let $O\xi$ be the portion of the radius vector intercepted by this sphere. The plane through $O\xi$ and Oz intercepts the sphere in the great circle $q\xi z\zeta$. Through $O\xi$ pass a plane perpendicular to the $\xi\zeta$-plane. This is the $\xi\eta$-plane. It intercepts the sphere in the great circle $\xi\eta$. If $O\xi$ is the ξ-axis and $O\zeta$ is the ζ-axis, then $O\eta$ is the η-axis and it lies in the intersection of the xy- and the $\xi\eta$-planes. Then draw the great circles, $x\xi$, ξy, $y\zeta$, ηz, and $x\zeta$.

In the spherical triangle $x\xi q$, the side xq is the longitude θ and the side $q\xi$ is the latitude φ. The side $x\xi$ is the angle whose cosine is α_1, since it is the angle between the x- and the ξ-axes. The angle opposite the side $x\xi$ is a right angle. Then the first equation of Eq. (1) gives

$$\alpha_1 = \cos \varphi \cos \theta.$$

In the triangle $\xi q y$, qy is equal to $90° - \theta$, and ξy is the angle whose cosine is β_1. The first of Eq. (1) applied to this triangle gives

$$\beta_1 = \cos\varphi \sin\theta.$$

The arc ξz is $90° - \varphi$, and is the angle whose cosine is γ_1. Hence

$$\gamma_1 = \sin\varphi.$$

Since the arc $q\eta$ is equal to $90°$ (η is the pole of the great circle $\zeta z \xi q$), the arc $x\eta$ is equal to $90° + \theta$ and since $x\eta$ is the angle whose cosine is α_2 and $y\eta$ is the angle whose cosine is β_2,

$$\alpha_2 = -\sin\theta,$$
$$\beta_2 = +\cos\theta.$$

The arc $z\eta$ is the angle whose cosine is γ_2, but since $z\eta$ is equal to $90°$

$$\gamma_2 = 0.$$

The arc $z\zeta$ is obviously equal to φ, and since $z\zeta$ is the angle whose cosine is γ_3

$$\gamma_3 = \cos\varphi.$$

In the spherical triangle $xz\zeta$ the side $x\zeta$ is the angle whose cosine is α_3, xz is equal to $90°$, and $z\zeta$ is equal to φ. The angle included between the two sides xz and $z\zeta$ is $\pi - \theta$. Hence, the first equation of Eq. (1) gives

$$\alpha_3 = -\sin\varphi\cos\theta.$$

Finally, in the triangle $yz\zeta$, the side yz is $90°$, the side $z\zeta$ is φ, and the side $y\zeta$ is the angle whose cosine is β_3. The angle between the two sides yz and $z\zeta$ is $90° + \theta$. Hence, the first equation of Eq. (1) gives

$$\beta_3 = -\sin\varphi\sin\theta.$$

Hence, the table for the direction cosines as functions of the polar angles is

$$\left.\begin{aligned}
\alpha_1 &= \cos\varphi\cos\theta, & \alpha_2 &= -\sin\theta, & \alpha_3 &= -\sin\varphi\cos\theta,\\
\beta_1 &= \cos\varphi\sin\theta, & \beta_2 &= +\cos\theta, & \beta_3 &= -\sin\varphi\sin\theta,\\
\gamma_1 &= \sin\varphi, & \gamma_2 &= 0, & \gamma_3 &= +\cos\varphi.
\end{aligned}\right\} \quad (2)$$

262. The Components of Velocity in Spherical Coordinates.—On differentiating Eq. (260.1) it is found that

$$\left.\begin{aligned}
v_x &= x' = r'\cos\varphi\cos\theta - r\varphi'\sin\varphi\cos\theta - r\theta'\cos\varphi\sin\theta,\\
v_y &= y' = r'\cos\varphi\sin\theta - r\varphi'\sin\varphi\sin\theta + r\theta'\cos\varphi\cos\theta,\\
v_z &= z' = r'\sin\varphi + r\varphi'\cos\varphi + 0.
\end{aligned}\right\} \quad (1)$$

The substitution of Eq. (261.2) and (1) in Eq. (260.3) gives the desired equations

$$\left.\begin{aligned} v_r &= r', \\ v_\varphi &= r\varphi', \\ v_\theta &= r\theta' \cos\varphi. \end{aligned}\right\} \tag{2}$$

263. The Component of Acceleration in Spherical Coordinates.—By differentiation of Eq. (262.1), the following is obtained:

$$\left.\begin{aligned} x'' &= [r'' - r\varphi'^2 - r\theta'^2 \cos^2\varphi] \cos\varphi \cos\theta \\ &\quad - [r\varphi'' + 2r'\varphi' + r\theta'^2 \sin\varphi \cos\varphi] \sin\varphi \cos\theta \\ &\quad - [r\theta'' \cos\varphi - 2r\varphi'\theta' \sin\varphi + 2r'\theta' \cos\varphi] \sin\theta, \\ y'' &= [r'' - r\varphi'^2 - r\theta'^2 \cos^2\varphi] \cos\varphi \sin\theta \\ &\quad - [r\varphi'' + 2r'\varphi' + r\theta'^2 \sin\varphi \cos\varphi] \sin\varphi \sin\theta \\ &\quad + [r\theta'' \cos\varphi - 2r\varphi'\theta' \sin\varphi + 2r'\theta' \cos\varphi] \cos\theta, \\ z'' &= [r'' - r\varphi'^2 - r\theta'^2 \cos^2\varphi] \sin\varphi \\ &\quad + [r\varphi'' + 2r'\varphi' + r\theta'^2 \sin\varphi \cos\varphi] \cos\varphi. \end{aligned}\right\} \tag{1}$$

On substituting Eqs. (261.2) and (1) in Eq. (260.3), the components of the acceleration along the radius vector and in the two perpendicular directions which correspond to latitude and longitude, expressed in terms of the spherical coordinates and their derivatives, are

$$\left.\begin{aligned} \alpha_r &= r'' - r\varphi'^2 - r\theta'^2 \cos^2\varphi, \\ \alpha_\varphi &= r\varphi'' + 2r'\varphi' + r\theta'^2 \sin\varphi \cos\varphi, \\ \alpha_\theta &= r\theta'' \cos\varphi - 2r\varphi'\theta' \sin\varphi + 2r'\theta' \cos\varphi. \end{aligned}\right\} \tag{2}$$

Another method of deriving these accelerations is given in Sec. 352.

264. Uniform Motion in a Circle.—If a particle moves in a circle of radius a with constant speed, the angle which it describes is proportional to the time, that is,

$$\theta = \omega t,$$

where ω, which is constant, is the angular velocity. Then it follows that

$$x = a \cos\omega t, \qquad y = a \sin\omega t;$$

from which are derived

$$\begin{aligned} x' &= -a\omega \sin\omega t, & y' &= +a\omega \cos\omega t, \\ x'' &= -a\omega^2 \cos\omega t, & y'' &= -a\omega^2 \sin\omega t. \end{aligned}$$

Hence,

$$\begin{aligned} v &= \sqrt{x'^2 + y'^2} = a\omega, \\ \alpha &= \sqrt{x''^2 + y''^2} = a\omega^2. \end{aligned}$$

Since

$$x'' = -\omega^2 x, \qquad y'' = -\omega^2 y,$$

it is evident that the acceleration is always directed toward the center of the circle (Sec. 39), and is constant in magnitude, namely,

$$\alpha = a\omega^2 = \frac{v^2}{a}.$$

The hodograph is a circle of radius $a\omega$, and the position in the hodograph is 90° ahead of the position of the particle in its circle (see Fig. 21). The velocity in the hodograph is parallel to the radius vector of the particle, but is oppositely directed. The velocity in the hodograph is, of course, the acceleration of the particle.

265. The Effect of the Rotation of the Earth on the Acceleration of Gravity.—It will be assumed that the earth is a sphere which is homogeneous in concentric layers so that the attraction of the earth is directed toward the center of the earth. In Fig. 145, let QB be a light string which is supporting a heavy sphere B at rest relative to the surface of the earth, so that QB is the plumb line at B.

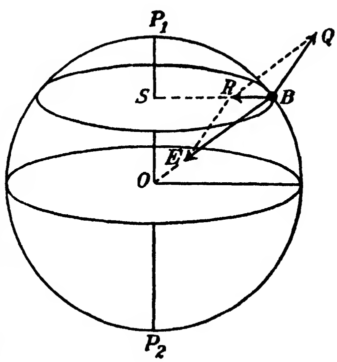


FIG. 145.

Relative to a set of rectangular axes with its origin at the center of the earth, the z-axis coinciding with the axis of the rotation of the earth and the x- and y-axes fixed in direction relative to the stars, the weight B describes a circle of radius BS with the constant angular speed ω. Its acceleration α, therefore, is directed toward the point S, the center of the circle which it describes, and in magnitude

$$\alpha = \overline{BS}\omega^2 = r\omega^2 \cos l,$$

where r is the radius of the earth, and l is the latitude of B.

The acceleration α, however, is the resultant of two accelerations to which B is subject; one is the acceleration $\mathbf{G}$ directed toward the center of the earth, due to the attraction of the earth, and the other is $-\mathbf{g}$, directed along the string toward the point of suspension, and is due to the tension of the string. Let

these accelerations be represented by the sides of the triangle *BRE*, in which the angle at B is the latitude l of B. Then by the cosine law,

$$\begin{aligned} g^2 &= G^2 - 2\alpha G \cos l + \alpha^2 \\ &= G^2 - 2Gr\omega^2 \cos^2 l + r^2\omega^4 \cos^2 l. \end{aligned}$$

Since there are 86,164.1 seconds in a sidereal day,

$$\omega = \frac{2\pi}{86{,}164.1} = 0.000072921 \quad [\log \omega = 5.862854 - 10]$$

also

$$r = 20{,}900{,}000 \text{ ft.} \qquad [\log r = 7.320149].$$

Hence,

$$r\omega^2 = 0.11114.$$

Since $r\omega^2/G$ is small (approximately 1/300), the expression for g, namely,

$$g = G\left[1 - 2\frac{r\omega^2}{G} \cos^2 l + \frac{r^2\omega^4}{G^2} \cos^2 l\right]^{\frac{1}{2}},$$

can be expanded in powers of $r\omega^2/G$ by the binomial theorem with the result

$$\begin{aligned} g &= G\left[1 - \frac{r\omega^2}{G} \cos^2 l + \cdots\right] \\ &= G\left(1 - \frac{r\omega^2}{2G}\right) - \frac{1}{2}r\omega^2 \cos 2l + \cdots \\ &= 32.160 - 0.056 \cos 2l. \end{aligned}$$

The earth, however, is not a sphere but is an oblate spheroid, so that the attraction of the earth is not toward its center, the direction varying with the latitude. These figures, therefore, do not represent the entire situation. If G is taken to be the actual acceleration of the earth at sea level relative to a set of axes at the center of the earth as described above and g is the acceleration relative to a set of axes fixed on the surface of the earth the correct equations are

$$\left.\begin{aligned} G &= 32.225 - .026 \cos 2l, \\ g &= 32.174 - .085 \cos 2l. \end{aligned}\right\} \tag{1}$$

266. Motion in an Ellipse.—If the position of a particle is given by the equations

$$x = a \cos nt, \qquad y = b \sin nt,$$

the path, which is obtained by eliminating the time between these two equations, is the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

It is found by differentiation that

$$x' = -an \sin nt, \qquad x'' = -an^2 \cos nt = -n^2x,$$
$$y' = +bn \cos nt, \qquad y'' = -bn^2 \sin nt = -n^2y.$$

Since the components of the acceleration are proportional to the coordinates, the total acceleration is

$$\alpha = -n^2\mathbf{r},$$

and it is therefore always directed toward the origin.

The equation of the hodograph is

$$\frac{x'^2}{a^2n^2} + \frac{y'^2}{b^2n^2} = 1,$$

which is the equation of an ellipse similar to the path itself, but of different size unless n^2 is equal to unity.

267. The Motion of a Particle on the Circumference of a Rolling Wheel.—If a wheel of radius a rolls along a straight

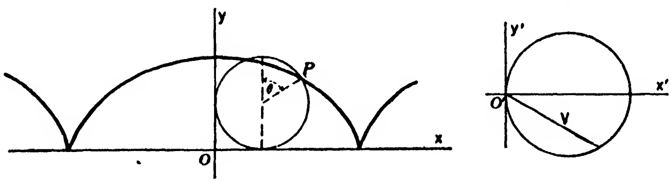


FIG. 146.

line, a particle on its circumference describes a cycloid, of which the parametric equations are

$$\left.\begin{aligned} x &= a(\theta + \sin \theta), \\ y &= a(1 + \cos \theta), \end{aligned}\right\} \qquad (1)$$

where θ is the angle which the particle makes at the center with the highest point of the circle. If the wheel rolls with a constant angular speed, the angle θ will be equal to ωt, where ω is some constant. Hence,

$$x' = a\omega(1 + \cos \omega t), \qquad x'' = -a\omega^2 \sin \omega t,$$
$$y' = -a\omega \sin \omega t, \qquad y'' = -a\omega^2 \cos \omega t.$$

The acceleration is constant in magnitude and equal to $a\omega^2$. It is always directed toward the center of the circle. This result

could have been anticipated, for the motion of the wheel is the sum of a uniform rotation about its center, in which the acceleration is constant and always directed toward the center, and a uniform translation in which the acceleration is zero.

The equation of the hodograph is

$$(x' - a\omega)^2 + y'^2 = a^2\omega^2. \tag{2}$$

The curve is therefore a circle through the origin. Its center is on the x'-axis and its radius is equal to $a\omega$.

268. The Equations of Motion When the Force Is Given.—In the preceding examples, the coordinates of the particle were given as functions of the time, and the velocity and acceleration determined by the process of differentiation, which is relatively simple. If, however, the law of the force under which the particle is moving is given, the problem is inverted, and the process of integration is required. In general, the law of the force is a function of the coordinates of the particle; the equations of motion are differential equations of the second order, and there is one such equation for each coordinate of the particle.

Let X, Y, and Z be the x-, y-, and z-components, respectively, of the resultant force which is acting upon the particle, and let m be the mass of the particle. Then, in accordance with Newton's second law, the equations of motion are

$$\left.\begin{aligned} mx'' &= X, \\ my'' &= Y, \\ mz'' &= Z. \end{aligned}\right\} \tag{1}$$

If there exists a force function (potential function Sec. 64), $U(x, y, z)$, such that

$$X = \frac{\partial U}{\partial x}, \qquad Y = \frac{\partial U}{\partial y}, \qquad Z = \frac{\partial U}{\partial z},$$

the equations of motion are

$$mx'' = \frac{\partial U}{\partial x}, \qquad my'' = \frac{\partial U}{\partial y}, \qquad mz'' = \frac{\partial U}{\partial z}. \tag{2}$$

269. The Energy Integral.—On multiplying the first equation of Eq. (268.1) by dx, the second by dy, the third by dz, and adding, there is obtained the equation

$$\begin{aligned} m(x''dx + y''dy + z''dz) &= d\left[\frac{1}{2}m(x'^2 + y'^2 + z'^2)\right] \\ &= Xdx + Ydy + Zdz. \end{aligned} \tag{1}$$

It is seen from Sec. 58 that the right member of Eq. (1) is the work done by the force acting in the displacement ds. The left member is the change in the kinetic energy $mv^2/2$. Equation (1), therefore, states that the change in the kinetic energy of the particle is equal to the work done upon the particle by the force which is acting in any infinitesimal displacement ds. Since

$$v^2 = x'^2 + y'^2 + z'^2,$$

the integration of Eq. (1) gives

$$\frac{1}{2}mv^2 - \frac{1}{2}mv_0^2 = \int_{x_0,y_0,z_0}^{x,y,z} [Xdx + Ydy + Zdz], \tag{2}$$

the integral of the right member to be computed along the path followed by the particle.

If $Xdx + Ydy + Zdz$ is not an exact differential, the right member of Eq. (2) will depend not only on the points x_0, y_0, z_0 and x, y, z, but also upon the path which the particle pursues between them.

If a potential function $U(x, y, z)$ exists, then (Sec. 64)

$$Xdx + Ydy + Zdz = dU$$

is an exact differential, and the integral Eq. (2) becomes

$$\frac{1}{2}mv^2 - \frac{1}{2}mv_0^2 = U(x, y, z) - U(x_0, y_0, z_0), \tag{3}$$

which is independent of the path from x_0, y_0, z_0 to x, y, z.

Equation (3) can be written also

$$\frac{1}{2}mv^2 - U(x, y, z) = E \text{ (a constant)}, \tag{4}$$

where

$$\frac{1}{2}mv^2 = \text{the kinetic energy},$$

$$-U(x, y, z) = \text{the potential energy},$$

and

$$E = \text{the total energy}.$$

Thus, when there exists a potential function, the force is conservative, the energy of the particle is constant, and Eq. (4) is called the *energy integral.*

270. The Motion of a Projectile in Vacuo.—If the distances considered are not too great, the attraction of the earth on a particle is virtually constant and parallel to any given vertical in its vicinity. The horizontal component of the force which is

acting is zero. Let the projectile be fired a certain direction, and let the plane which is determined by the vertical and the initial velocity be taken as the xz-plane, the origin being taken at the gun. With the positive end of the z-axis directed upward and the positive end of the x-axis in the direction of fire, the equations of motion are

$$mx'' = 0, \qquad my'' = 0, \qquad mz'' = -mg; \tag{0}$$

and the initial conditions are

$$\text{at } t = 0, \qquad \begin{aligned} x = y = z = y' &= 0, \\ x' &= v_0 \cos \alpha, \\ z' &= v_0 \sin \alpha. \end{aligned}$$

It will be observed that the factor m can be removed from the equations (0). The motion, therefore, does not depend upon the mass of the particle. The integration of the second equation gives the information that y' is constant. Since it is zero initially, it is always zero and therefore y is constant. Since y is initially zero, it is zero throughout the motion and the path of motion lies in the xz-plane.

On removing the factor m from each of the first and third equations and then integrating each equation once, it is found that

$$x' = c_1, \qquad z' = -gt + c_2,$$

where c_1 and c_2 are the constants of integration. Since, however,

$$x' = v_0 \cos \alpha, \qquad z' = v_0 \sin \alpha,$$

for $t = 0$, it follows that

$$c_1 = v_0 \cos \alpha, \qquad c_2 = v_0 \sin \alpha;$$

and, therefore,

$$x' = v_0 \cos \alpha, \qquad z' = -gt + v_0 \sin \alpha. \tag{1}$$

If each of these equations is integrated again and the constants of integration determined so as to satisfy the initial conditions, there results

$$x = v_0 \cos \alpha \cdot t, \qquad z = -\frac{1}{2}gt^2 + v_0 \sin \alpha \cdot t. \tag{2}$$

The elimination of t between these two equations gives the equation of the path, namely,

$$z = x \tan \alpha - \frac{gx^2}{2v_0^2 \cos^2 \alpha},$$

which is a parabola (Fig. 147).

Let the constant h be introduced by the relation

$$v_0^2 = 2gh.$$

Then the equation of the parabola becomes

$$z = x \tan \alpha - \frac{x^2}{4h \cos^2 \alpha}. \tag{3}$$

By means of the substitutions,

$$\xi = x - 2h \sin \alpha \cos \alpha, \qquad \zeta = z - h \sin^2 \alpha,$$

Eq. (3) is simplified, and becomes

$$\zeta = -\frac{\xi^2}{4h \cos^2 \alpha}, \tag{4}$$

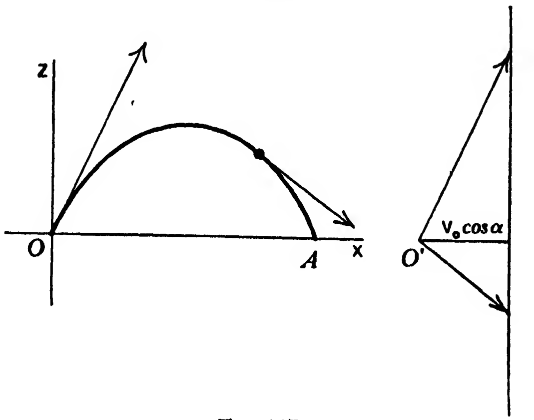


FIG. 147.

from which it is evident that the axis of the parabola is vertical with the vertex upward, and that the coordinates of the vertex with respect to the gun are

$$\bar{x} = h \sin 2\alpha, \qquad \bar{z} = h \sin^2 \alpha. \tag{5}$$

Evidently, h is the maximum height for all values of α which can be attained by the projectile for a given v_0 and this value is attained for α equal to 90°.

The coordinates of the focus of the parabola are

$$x_f = h \sin 2\alpha, \qquad z_f = -h \cos 2\alpha.$$

Hence, the focus lies on the circle with the origin as center and a radius equal to h.

The locus of the vertex as α varies is obtained by eliminating α between the two equations of Eq. (5). Its equation is

$$\bar{x}^2 + (2\bar{z} - h)^2 = h^2, \tag{6}$$

which represents an ellipse through the origin, the center being at a distance equal to $h/2$ above the origin, with the horizontal axis twice the vertical axis.

If α is eliminated between the two equations of Eq. (2), it is seen that whatever value α may have the projectile lies on the circle

$$x^2 + (z + \frac{1}{2}gt^2)^2 = v_0^2t^2. \tag{7}$$

Hence, if many projectiles were fired simultaneously from the same point, but at different angles of elevation, the projectiles

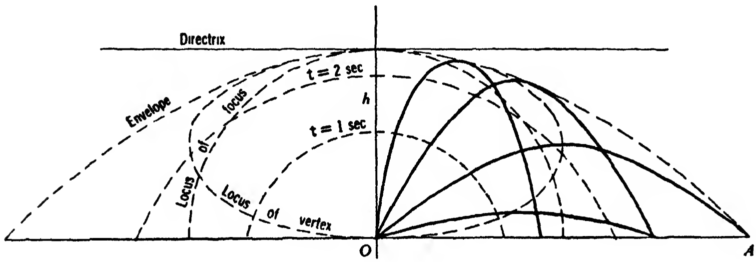


FIG. 148.

would at each instant lie on a circle (the *synchronous curve*) the radius of which would be v_0t.

The equation

$$F(x, z, \alpha) = 0, \tag{8}$$

in which α is regarded as a parameter, represents a family of curves. The envelope of this family of curves is obtained by eliminating α between Eq. (8) and the equation

$$\frac{dF(x, z, \alpha)}{d\alpha} = 0. \tag{9}$$

In this way the envelope of the family of parabolas, represented by Eq. (3), is found to be

$$x^2 + 4hz - 4h^2 = 0, \tag{10}$$

which is a parabola with its vertex on the z-axis at a height h above the origin.

The equations of the hodograph are given parametrically (the parameter being t) by Eq. (1). It is seen that the hodograph is a vertical straight line, which is obvious otherwise, since the acceleration is always vertical (Fig. 147).

The directrix of the parabola (Eq. (3)) is the straight line

$$z = h;$$

and, since this equation is independent of α, the same straight line is the directrix for all the parabolas which have the same initial speed.

If the first equation of Eq. (0) is multiplied by $2x'$ and then integrated and the third is multiplied by $2z'$ and integrated, the results added together give the velocity integral

$$x'^2 + z'^2 = v_0^2 - 2gz, \tag{11}$$

provided the constant of integration is determined so as to satisfy the initial conditions. From this integral, the speed can be determined at any point of the curve.

Let x, z be a point on the curve, and let P be a point on the directrix directly above it. Imagine a particle dropped from rest at P. It will pass through the point x, z after having fallen through the distance $h - z$. By Eq. (239.5), its speed at the point x, z is

$$v^2 = 2g(h - z) = v_0^2 - 2gz,$$

which is the same as Eq. (11). The speed of the projectile, therefore, at any point in the parabola is the same as its speed would have been had it dropped from rest from the point on the directrix directly above. The velocity would be different, for the direction of motion would be different.

The horizontal range of the projectile is the distance OA in Fig. 147. Its value is obtained by setting z equal to zero in Eq. (3), and then solving for x. The solution x equal to zero corresponds to the position of the gun. From the other value of x, it is found that

$$\text{the horizontal range} = 2h \sin 2\alpha.$$

For a given initial speed, the horizontal range is a maximum if the initial angle α is 45°. For any other given horizontal range, there are two angles of elevation, namely,

$$\alpha = 45° - \beta, \qquad \alpha = 45° + \beta,$$

where β is an angle which depends upon the given range.

Indeed it is evident that there are two trajectories through any given point within the envelope, but only one if the point is on the envelope and none if the point is outside of the envelope. If

the point x, z is assumed to be given and one seeks the value of the angle α, Eq. (3) takes the form

$$x^2 \tan^2 \alpha - 4hx \tan \alpha + (x^2 + 4hz) = 0;$$

and since this is an equation of the second degree in $\tan \alpha$ there are two real solutions, a double solution, or no real solution, according as the discriminant

$$16h^2x^2 - 4x^2(x^2 + 4hz) = 4x^2[4h^2 - 4hz - x^2] \gtreqless 0.$$

The limiting condition

$$x^2 + 4hz - 4h^2 = 0,$$

which separates the real and complex solutions, is the same as Eq. (10), the equation of the envelope. Thus the points which lie outside of the envelope could be reached only by increasing the initial speed.

271. The Effect of a Resisting Medium on a Projectile.—A projectile moving in a vacuum can be regarded as a particle, since the acceleration due to gravity is the same on every particle of the projectile. But if a projectile is moving in a medium such as the air, the surface of the projectile is acted upon by the pressure of the medium, and the nature of that pressure varies in a very complicated way with the nature of the medium, the shape of the projectile, the distribution of the mass within it, its orientation, its velocity, its rate of spin, and its axis of spin. The problem is an extremely complicated one and forms the subject matter of exterior ballistics.[1]

If, however, the projectile is a homogeneous sphere not rotating, and the resisting medium is at rest, the projectile can be regarded as a particle which is acted upon by a force which in magnitude is some function of the speed of the projectile and which has a direction opposite to the velocity of the projectile. Since the resistance of the medium is a force which lies in the vertical plane of motion, it does not alter the initial plane of motion which remains the plane of motion throughout. If θ is the angle which the velocity of the projectile makes with the x-axis, the forces which are acting on the projectile resolved along the tangent to the path, and normal to it, are

$$f_t = -mg \sin \theta - R,$$
$$f_n = +mg \cos \theta.$$

[1] See MOULTON, F. R., "New Methods in Exterior Ballistics," 1926.

Since the resistance R is a tangential force, it does not alter the direction of motion. Since the force of gravity is the only force which is not tangential, the trajectory is always concave downward, and, therefore, the component of gravity along the normal on the concave side is always positive. The component of gravity along the tangent changes sign with θ.

The intrinsic equations of motion (Eq. (258.1)) are then

$$mv' = -mg \sin \theta - R,$$

$$m\frac{v^2}{\rho} = mg \cos \theta.$$

Since R the resistance is a function of v, it can be written

$$R = mg\varphi(v);$$

also

$$\rho = -\frac{ds}{d\theta} = -\frac{ds}{dt}\cdot\frac{dt}{d\theta} = -v\frac{dt}{d\theta}.$$

The negative sign is taken since s increases as θ decreases.

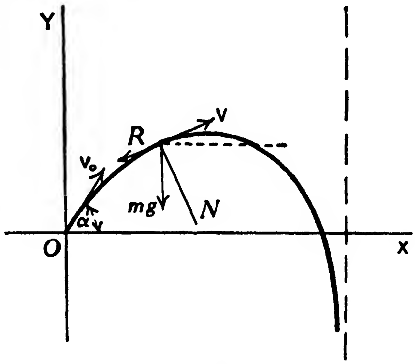


FIG. 148a.

On substituting these values of R and ρ and then removing the factor m, the equations of motion become

$$v' = -g(\sin \theta + \varphi(v)), \qquad v\theta' = -g \cos \theta; \tag{1}$$

and the time t can be eliminated between these two equations by taking their quotient, namely,

$$\frac{1}{v}\frac{dv}{d\theta} = \tan \theta + \varphi(v) \sec \theta, \tag{2}$$

a differential equation of the first order between v and θ.

If the solution of this equation,

$$v = f(\theta)$$

were known, the second equation of Eq. (1) would give the time t by a quadrature,

$$t = -\frac{1}{g}\int_{\theta_0}^{\theta} f(\theta) \sec \theta \, d\theta. \tag{3}$$

By virtue of the fact that

$$x' = \frac{dx}{ds}\cdot\frac{ds}{dt} = v \cos \theta,$$

and

$$y' = \frac{dy}{ds}\cdot\frac{ds}{dt} = v \sin \theta,$$

the x- and y-coordinates also would be determined by quadratures, namely,

$$x = -\frac{1}{g}\int_{\theta_0}^{\theta} f^2(\theta)d\theta \tag{4}$$

and

$$y = -\frac{1}{g}\int_{\theta_0}^{\theta} f^2(\theta) \tan \theta d\theta.$$

Thus the entire problem would be reduced to quadratures if the solution of Eq. (2) were known.

272. The Integrable Case of Legendre.—The nature of the function $\varphi(v)$ is unknown, but it vanishes with v and quite likely it always increases when v increases, that is, $d\varphi/dv$ is positive for all positive values of v. It was shown by Legendre that the equation can be integrated if

$$\varphi(v) = \frac{1}{n}(a + bv^n),$$

where a and b are two constants.

It will simplify the differential equation Eq. (271.2) somewhat if the independent variable is changed from θ to ω by the substitution

$$\sin \theta = \tanh \omega.$$

By this substitution Eq. (271.2) becomes

$$\frac{1}{v}\frac{dv}{d\omega} = \varphi(v) + \tanh \omega. \tag{1}$$

For Legendre's case, Eq. (1) becomes

$$\frac{n}{v}\frac{dv}{d\omega} = a + bv^n + n \tanh \omega;$$

and by the substitution

$$v^n = \frac{1}{u},$$

the equation becomes the linear equation

$$\frac{du}{d\omega} + (a + n \tanh \omega)u = -b.$$

The integrating factor for this equation, which is a well-known form, is

$$e^{+\int(a+n \tanh \omega)d\omega} = e^{+a\omega} \cosh^n \omega.$$

Therefore, the solution is

$$u = -e^{-\int (a+n\tanh\omega)d\omega}\int_{\omega_0}^{\omega} be^{+\int(a+n\tanh\omega)d\omega}\,d\omega.$$

$$= -e^{-a\omega}\operatorname{sech}^n\omega\int_{\omega_0}^{\omega} be^{+a\omega}\cosh^n\omega d\omega. \qquad (2)$$

For n equal to 2, this becomes

$$u = \frac{b}{4 - a^2}(a - 2\tanh\omega) + \frac{2b\operatorname{sech}^2\omega}{a(4 - a^2)} + Ce^{-a\omega}\operatorname{sech}^2\omega$$

where C is the constant of integration.

As the angle θ diminishes and approaches the limit $-\pi/2$, ω also diminishes and tends toward $-\infty$. Under these conditions,

$$\lim_{\omega=-\infty}\tanh\omega = -1, \qquad \lim_{\omega=-\infty}\operatorname{sech}\omega = 0,$$

and

$$\lim_{\omega=-\infty} e^{-a\omega}\operatorname{sech}^2\omega = \lim_{\omega=-\infty}\frac{4e^{-a\omega}}{\left(e^{\omega}+e^{-\omega}\right)^2} = 0, \qquad \text{if } a < 2.$$

Therefore, the limit of u is $\dfrac{b}{2 - a}$, and

$$\lim v^2 = \frac{2 - a}{b}. \qquad (3)$$

It is found by means of the reduction formula that in the general case

$$u = -be^{-a\omega}\operatorname{sech}^n\omega\int e^{a\omega}\cosh^n\omega\,d\omega = \frac{b}{n^2 - a^2}[a - n\tanh\omega]$$

$$-\frac{bn(n-1)}{(n^2 - a^2)}e^{-a\omega}\operatorname{sech}^n\omega\int e^{a\omega}\cosh^{n-2}\omega\,d\omega. \qquad (4)$$

The limit of the last term evidently is zero and therefore, in general, provided $a < n$,

$$\lim_{\omega=-\infty} u = \frac{b}{n - a}$$

and

$$\lim_{\omega=-\infty} v^n = \frac{n - a}{b},$$

and the speed tends toward a constant value.

For $a > n$, the first equation of Eq. (271.1) shows that if the projectile were abandoned in a position of rest, v' would be negative, which, of course, is absurd. It will be observed that the limiting value of the speed is a root of the equation $\varphi(v) = 1$.

The integral for the time t (Eq. (271.3)) becomes

$$t = -\frac{1}{g}\int_{\omega_0}^{\omega} u^{\frac{1}{n}} d\omega.$$

Since u tends toward a finite limit, the time t increases without limit as ω tends toward minus infinity.

The expressions of Eq. (271.4) for x and y become

$$x = -\frac{1}{g}\int_{\omega_0}^{\omega} u^{\frac{2}{n}} \operatorname{sech} \omega \, d\omega$$

and

$$y = -\frac{1}{g}\int_{\omega_0}^{\omega} u^{\frac{2}{n}} \tanh \omega \, d\omega.$$

Since u has a finite limit, $\tanh \omega$ has -1 as a limit and $\operatorname{sech} \omega$ tends to zero like $e^{-\omega}$ it is evident that x has a finite limit while y has not. Thus the trajectory is asymptotic to the vertical line which passes through the limiting value of x (Fig. 148a).

273. Analogy between Trajectories and Catenaries.—The differential equations for catenaries and trajectories, given in Eqs. (202.1) and (259.5), are

$$\begin{array}{ll} \text{For Catenaries} & \text{For Trajectories} \\ \dfrac{d}{ds}\left(T\dfrac{dx}{ds}\right) = -\sigma X, & \dfrac{d}{dt}\left(v\dfrac{dx}{ds}\right) = X_a, \\ \dfrac{d}{ds}\left(T\dfrac{dy}{ds}\right) = -\sigma Y, & \dfrac{d}{dt}\left(v\dfrac{dy}{ds}\right) = Y_a, \\ \dfrac{d}{ds}\left(T\dfrac{dz}{ds}\right) = -\sigma Z, & \dfrac{d}{dt}\left(v\dfrac{dz}{ds}\right) = Z_a, \end{array} \qquad (1)$$

where X, Y, and Z are the components of the force which is acting upon the chain per unit mass, σ the mass of the chain per unit length, and T the tension of the chain. X_a, Y_a, and Z_a, are the components of acceleration and v the velocity of the moving particle. These two sets of equations are strikingly similar.

Imagine a particle moving along the catenary with a speed which is always equal to the tension. Then the equations of the catenary become

$$\frac{d}{dt}\left(v\frac{dx}{ds}\right) = -\sigma TX = X_a,$$
$$\frac{d}{dt}\left(v\frac{dy}{ds}\right) = -\sigma TY = Y_a,$$
$$\frac{d}{dt}\left(v\frac{dz}{ds}\right) = -\sigma TZ = Z_a;$$

which show that the acceleration of the particle lies in the same line as the force which is acting upon the chain, but is oppositely directed. The magnitude of the acceleration is σT times the magnitude of the force, and is therefore variable. If $\mathbf{F}_1$ is the force which acts upon a unit particle in motion and $\mathbf{F}$ is the force which acts upon a unit mass of the chain, then

$$\mathbf{F}_1 = -\sigma T\mathbf{F}. \tag{2}$$

If the field of force is a conservative one, so that

$$X = \frac{\partial V}{\partial x}, \qquad Y = \frac{\partial V}{\partial y}, \qquad Z = \frac{\partial V}{\partial z},$$

and if σ is a constant, the tension of the chain is given by Eq. (203.1)

$$T = \sigma(V_0 - V).$$

Now let

$$U = \frac{1}{2}\sigma^2(V_0 - V)^2 = \frac{1}{2}T^2. \tag{3}$$

Then

$$\frac{\partial U}{\partial x} = -\sigma^2(V_0 - V)\frac{\partial V}{\partial x} = -\sigma TX,$$

$$\frac{\partial U}{\partial y} = -\sigma^2(V_0 - V)\frac{\partial V}{\partial y} = -\sigma TY,$$

$$\frac{\partial U}{\partial z} = -\sigma^2(V_0 - V)\frac{\partial V}{\partial z} = -\sigma TZ;$$

and the equations of motion of the particle can be written

$$x'' = \frac{\partial U}{\partial x}, \qquad y'' = \frac{\partial U}{\partial y}, \qquad z'' = \frac{\partial U}{\partial z}, \tag{4}$$

subject to the condition that

$$x'^2 + y'^2 + z'^2 = v^2 = 2U.$$

274. The Curve is the Ordinary Catenary.—The curve formed by a uniform chain under the action of gravity is the ordinary catenary (Sec. 192). The force acting is a conservative one and σ is constant. Therefore

$$V = -gz$$

and

$$T = \sigma gz,$$

if the vertical axis is the z-axis. Accordingly,

$$U = \frac{1}{2}\sigma^2 g^2 z^2.$$

If the plane of motion is the xz-plane, the equations of motion are

$$x'' = 0 = \frac{\partial U}{\partial x}$$

and

$$z'' = \sigma^2 g^2 z = \frac{\partial U}{\partial z}. \tag{1}$$

Let the initial point on the curve be

$$x = 0, \qquad z = z_0. \tag{2}$$

The general solution of the equations of Eq. (1) is

$$\begin{aligned} x &= c_1 t + c_2, \\ z &= c \cosh (\sigma g t + c_3); \end{aligned} \tag{3}$$

but if the curve passes through the point (2), then

$$\begin{aligned} c_2 &= 0 \\ z_0 &= c \cosh c_3. \end{aligned}$$

It is necessary also to satisfy the condition

$$x'^2 + z'^2 = \sigma^2 g^2 z^2 = 2U. \tag{4}$$

On differentiating Eq. (3) and substituting in Eq. (4), it is found that

$$c_1 = \pm c\sigma g.$$

Hence, the solution which satisfies all of the conditions is

$$\begin{aligned} \frac{x}{c} &= \sigma g t, \\ \frac{z}{c} &= \cosh (\sigma g t + c_3), \end{aligned}$$

which is the familiar equation of the catenary in parametric form. The force under which the particle is moving is repellant and its magnitude is proportional to the distance of the particle from the x-axis.

275. The Trajectory is a Parabola under the Action of Gravity. If a particle is thrown into the air at an angle α with the horizon and with a speed v_0, and if, as in Sec. 270,

$$v_0^2 = 2gh,$$

the trajectory is a parabola, and its equation, referred to its directrix as the x-axis, is

$$z + h \cos^2 \alpha = \frac{-x^2}{4h \cos^2 \alpha}.$$

The force of gravity is

$$F_1 = -g,$$

and the speed in the trajectory is

$$v = \sqrt{-2gz},$$

which is real, since z is always negative.

Instead of reversing the field, let the parabola be turned over by merely changing z into $-z$, leaving the field unaltered. Then the equation is

$$z - h\cos^2\alpha = \frac{x^2}{4h\cos^2\alpha},$$

which is to be regarded as a catenary under the action of a force which is everywhere vertical, but not necessarily constant. Since the tension in the catenary is always equal to the speed in the trajectory, the equation for the tension is

$$T = \sqrt{2gz}.$$

Therefore, by Eq. (273.2),

$$F\sigma = \frac{-g}{T} = \frac{-g}{\sqrt{2gz}}.$$

If the chain is uniform, that is, σ is constant, the force which is acting upon the chain is attractive and inversely proportional to the square root of the distance from the z-axis. But if the field is the gravitational field, then F is the same as F_1, namely $-g$, and σ varies,

$$\sigma = \frac{1}{\sqrt{2gz}}.$$

The weight of the chain between any two points is

$$W = g\int_{s_1}^{s_2}\sigma ds = g\int_{s_1}^{s_2}\frac{ds}{T}.$$

Since the x-component of the force is zero, the integration of the first equation of Eq. (273.1) gives

$$\frac{ds}{T} = \frac{dx}{T_0}$$

where T_0 is the tension at the vertex of the parabola, or its lowest point. Hence,

$$W = g\int_{s_1}^{s_2}\frac{ds}{T} = \frac{g}{T_0}\int_{x_1}^{x_2}dx$$
$$= g\frac{(x_2 - x_1)}{T_0}.$$

This equation shows that the weight of any arc of the parabola is proportional to the length of its projection upon the x-axis, a property already well known (Sec. 191).

Problems XX

1. If ξ, η is a set of rectangular axes rotating with the uniform angular speed ω with respect to the fixed axes x, y, show that

$$v_\xi = \xi' - \omega\eta, \qquad \alpha_\xi = \xi'' - 2\omega\eta' - \omega^2\xi,$$
$$v_\eta = \eta' + \omega\xi, \qquad \alpha_\eta = \eta'' + 2\omega\xi' - \omega^2\eta.$$

2. If a string will just support a weight of 100 lb. without breaking, how many times per second can a 1 lb. weight go around a circle of radius 1 ft. if it moves on a smooth horizontal surface and is constrained to the circle by the string, without the string breaking? *Ans.* 9.03.

3. A w-lb. weight attached to the end of a string describes a circle of 4000 miles radius in 24 hr. What is the tension in the string? *Ans.* $w/287$ lbs. approximately.

4. A particle is attached by a light string of length l to a fixed point. The particle describes a horizontal circle with the angular speed ω, the string making an angle α with the vertical. Show that

$$\cos\alpha = \frac{g}{l\omega^2}.$$

5. If the earth could maintain its shape and spin fast enough for objects at the equator to lose their weight entirely, what would be the length of the day? *Ans.* $1^h\ 25^m$ nearly.

6. A rubber band of weight w per unit length and modulus of elasticity λ is stretched to twice its natural length and placed around a horizontal pulley of radius a ft. How fast can the pulley spin without the band falling off?

Ans. $$\frac{1}{2\pi a}\sqrt{\frac{2\lambda g}{w}} \text{ r.p.s.}$$

7. A gun is fired from the foot of an inclined plane, inclination β. Show that the distance d along the inclined plane to the point where the projectile strikes, and the maximum value of this distance are, respectively

$$d = 4h\,\frac{\cos\alpha\sin(\alpha - \beta)}{\cos^2\beta},$$
$$\max = h\sec^2\left(\frac{\pi}{4} - \frac{\beta}{2}\right).$$

8. The maximum horizontal range of a gun is 20 miles. Show that to attain this range the initial speed is 1840 ft. per second, and that the time of flight is 81.3 sec.

9. Show that the range R of a projectile fired from a height H above the level ground is a root of the equation

$$R^2 - 2hR\sin 2\alpha - 4hH\cos^2\alpha = 0.$$

10. A gun is fired from an elevation H. What is the angle which gives the maximum horizontal range? *Ans.*

$$\tan \alpha = \sqrt{\frac{h}{h + H}}.$$

11. The base of a vertical regular hexagon of sides a is on a level plain A projectile passes through the four corners which are off the ground. Show that

$$h = \frac{31\sqrt{3}}{24}a, \qquad \text{horizontal range} = a\sqrt{7}.$$

12. A gun is fired with a muzzle velocity of 1000 ft. per second from the rear end of a train, which is traveling 45 miles per hour, in a direction opposite to that in which the train is moving. Show that the angle of elevation for the maximum range is

$$\alpha = 43° \, 37' \, 50''.$$

13. Show that if the bullet fired from a gun is projected from the point of fire upon a vertical plane the projected point moves with constant speed.

14. A bullet, fired at a vertical target at a distance a, strikes it at right angles. Show that the angle of elevation is $(\sin^{-1} a/h)/2$, and that the height of the point hit is one-half of the height of the point aimed at.

15. A gun is fired twice from the same point, the interval of time between the two shots being n seconds. Show that the two bullets will collide if the trajectories lie in the same plane, and

$$\frac{\sin \frac{1}{2}(\alpha_1 - \alpha_2)}{\cos \frac{1}{2}(\alpha_1 + \alpha_2)} = \frac{gn}{2v_0}.$$

16. Owing to unsteadiness, the direction of a gun may lie anywhere within a cone whose generating angle is θ and whose axis is the intended elevation α. If the initial speed v_0 is constant, and θ is so small that its powers higher than the first can be neglected, the bullets will all strike the ground within an ellipse whose semiaxes, a and b, in the plane of fire and perpendicular to it, respectively, are

$$a = 2h \sin 2\theta \cos 2\alpha, \qquad b = 2h \sin 2\theta \sin \alpha.$$

17. Work out the complete solution for a particle moving under the action of gravity in a resisting medium for which (Sec. 272) n is equal to 1 and a is equal to $1/2$.

18. If the resistance of the medium is proportional to the fourth power of the speed, show that the differential equation of the trajectory is

$$\frac{d^3y}{dx^3} = k\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}.$$

19. If the potential function for a free particle of mass m is

$$U(x, y, z) = k^2mxy,$$

show that the equations of motion can be completely integrated.

20. If a particle moves in the xy-plane and the components of force, X and Y, satisfy the relations

$$\frac{\partial X}{\partial y} = -\frac{\partial Y}{\partial x}, \qquad \frac{\partial X}{\partial x} = \frac{\partial Y}{\partial y},$$

show that the equations of motion can be integrated by two quadratures. Suggestion: Under these circumstances, $X + iY$ is a function of the complex variable $z = x + iy$. That is,

$$X + iY = \varphi(z).$$

The two equations can therefore be reduced to the single one

$$mz'' = \varphi(z),$$

the solution of which depends upon two quadratures.

CHAPTER XII

CENTRAL FORCES

I. GENERAL PROPERTIES

276. Definition of a Central Force.—Whenever the force which acts upon a particle is always directed toward a fixed point, wherever the particle may be, the force which is acting is said to be a *central force.* Its intensity may depend upon the position of the particle, its velocity, or even the time, but its direction is always in the straight line which passes through the particle and the fixed point, which is called the *center of force.* If the force is directed toward the fixed point it is an *attractive force.* If it is directed away from the fixed point it is a *repellant force.*

277. The Equations of Motion.—Let f be the magnitude of the force which is acting; f_x, f_y, and f_z its components parallel to the x-, y-, and z-axes; and λ, μ, and ν its direction cosines. Let the origin of the rectangular axes be at the center of force. Then

$$f_x = \lambda f, \qquad \lambda = \frac{x}{r},$$

$$f_y = \mu f, \qquad \mu = \frac{y}{r},$$

$$f_z = \nu f, \qquad \nu = \frac{z}{r}.$$

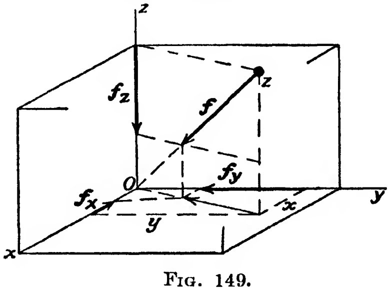


FIG. 149.

If the mass of the particle is m, the equations of motion are

$$\left.\begin{aligned} mx'' &= f\cdot\frac{x}{r}, \\ my'' &= f\cdot\frac{y}{r}, \\ mz'' &= f\cdot\frac{z}{r}. \end{aligned}\right\} \qquad (1)$$

If the force is an attractive force, f is negative. If it is a repellant force, f is positive.

The differential equations in Eq. (1) are of the sixth order, that is, three equations each of the second order; therefore, six

constants of integration are necessary for a complete solution. These six constants of integration can be regarded as being determined by the six initial coordinates of position and velocity, namely, x_0, y_0, z_0; x_0', y_0', and z_0'. Other interpretations, however, may be convenient and desirable.

278. The Moment of Momentum Integrals.—Multiplying the second equation of Eq. (277.1) by $-z$ and the third by $+y$ and then adding, the following equation is obtained:

$$m(yz'' - zy'') = 0,$$

and similarly,

$$m(zx'' - xz'') = 0,$$
$$m(xy'' - yx'') = 0.$$

These equations can be integrated immediately, with the result

$$\left.\begin{aligned} m(yz' - zy') &= mc_1, \\ m(zx' - xz') &= mc_2, \\ m(xy' - yx') &= mc_3, \end{aligned}\right\} \tag{1}$$

the right members of which are the constants of integration.

The components of the momentum of the particle, which is a vector, are

$$mx', \qquad my', \qquad mz'.$$

Therefore, by Sec. 132, the equations in Eq. (1) are the components of the *moment of momentum*, which, by Sec. 133, also is a vector. Since each of the three components is constant, the vector itself is constant. Thus the three equations in Eq. (1) can be summed up in the single statement: *If the force acting upon a particle is a central force, the moment of momentum of the particle is constant.* It will be observed that this theorem is independent of the law of the intensity of the force.

If the first equation of Eq. (1) is multiplied by x, the second by y, the third by z, and the three equations are then added, it is found that

$$c_1x + c_2y + c_3z = 0. \tag{2}$$

Since x, y, and z are the coordinates of the particle, it follows that *the particle always lies in a fixed plane which passes through the origin.*

279. The Energy Integral.—On multiplying the first equation of Eq. (277.1) by x', the second by y', the third by z', and then adding the three equations, it is found that

$$m(x'x'' + y'y'' + z'z'') = f \cdot \frac{xx' + yy' + zz'}{r} = f \cdot r', \tag{1}$$

since

$x^2 + y^2 + z^2 = r^2$

$$xx' + yy' + zz' = rr'.$$

The left member of Eq. (1) is an exact derivative, that is,

$$x'x'' + y'y'' + z'z'' = \frac{1}{2}(x'^2 + y'^2 + z'^2)'.$$

The right member also is an exact derivative if f is a function of r alone, that is, if f does not depend upon r', θ, θ', or t. In this event, there exists a potential function $U(r)$, such that

$$f(r) = \frac{\partial U}{\partial r},$$

and Eq. (1) can be integrated. The result of integration is

$$\frac{1}{2}m(x'^2 + y'^2 + z'^2) - U(r) = C. \tag{2}$$

The first term of Eq. (2), $mv^2/2$, is the kinetic energy of the particle. The second term $-U(r)$ is the potential energy. The integral Eq. (2), therefore, states that *the kinetic energy of the particle plus its potential energy is a constant,* which is here denoted by the letter C.

It will be observed that the energy integral exists only if f is a function of r alone, or can be expressed as a function of r alone. If this condition is not satisfied, the energy of the particle is not, in general, constant.

280. The Motion in the Plane.—That the motion of the particle lies in a plane Eq. (278.1) is evident intuitionally. A plane is determined by the position of the center of force, the initial position of the particle, and the initial direction of the motion of the particle. The force acting upon the particle also lies in this plane, so that there is nothing to make the particle depart from it.

Since the orientation of the axes of reference is arbitrary, it is simpler to choose the plane of motion of the particle as the xy-plane. Then

$$z \equiv 0, \qquad c_1 = c_2 = 0,$$

and the equations of motion reduce to the fourth order set

$$\left.\begin{aligned} mx'' &= f \cdot \frac{x}{r}, \\ my'' &= f \cdot \frac{y}{r}. \end{aligned}\right\} \tag{1}$$

The moment of momentum integral (Eq. (278.1)), after the removal of the factor m and the replacement of the letter c_3 by the letter h, becomes

$$xy' - yx' = h. \tag{2}$$

The energy integral, evidently, is

$$\frac{1}{2}m(x'^2 + y'^2) - U(r) = C. \tag{3}$$

281. The Equations of Motion in Polar Coordinates.—Using the expressions for the acceleration along the radius vector and perpendicular to it, given in Eq. (257.5), the equations of motion in polar coordinates, are

$$\left.\begin{aligned} m(r'' - r\theta'^2) &= f, \\ m(r\theta'' + 2r'\theta') &= \frac{m}{r}(r^2\theta')' = 0. \end{aligned}\right\} \tag{1}$$

It is, perhaps, well to recall once more that f is positive if the force is directed away from the center of force, that is, the force is repellant; and that f is negative if the force is attractive.

From the second equation of Eq. (1), it follows immediately that

$$r^2\theta' = h = xy' - yx'. \tag{2}$$

Since

$$ds^2 = dx^2 + dy^2 = dr^2 + r^2d\theta^2,$$

the energy integral (Eq. (280.3) becomes

$$\frac{1}{2}m(r'^2 + r^2\theta'^2) - U(r) = C. \tag{3}$$

282. The Linear Speed.—If r and θ are the polar coordinates of a point on a curve, p is the length of the perpendicular from the origin to the tangent of the curve, and s is the length of the arc, then, from the differential calculus,[1]

$$\frac{ds}{d\theta} = \frac{r^2}{p}.$$

Since

$$\frac{ds}{d\theta} = \frac{s'}{\theta'},$$

and, from Eq. (281.2),

$$\theta' = \frac{h}{r^2},$$

it follows that

$$s' = \frac{h}{p}. \tag{1}$$

[1] WILLIAMSON, "Differential Calculus," p. 224.

Since s' is the linear speed, Eq. (1) states that *if the force is a central force the speed of the particle is inversely proportional to the length of the perpendicular from the center of force to the tangent of the curve at the particle.*

283. The Areal Velocity.—Let A denote the area which has been swept over by the radius vector, starting from some convenient initial position, say $\theta = 0$ (Fig. 150).

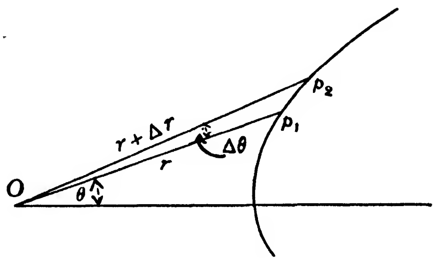


FIG. 150.

Let p_1 and p_2 be two positions of the particle which subtend an angle $\Delta\theta$ at the origin. Then, aside from terms which depend upon the curvature of arc p_1p_2, ΔA is equal to the area of the triangle Op_1p_2. Hence,

$$\Delta A = \frac{1}{2}(r + \Delta r) \cdot r \sin \Delta\theta + (\Delta\theta)^2[\ ],$$

and

$$\frac{\Delta A}{\Delta t} = \frac{1}{2}(r + \Delta r) \cdot r \frac{\sin \Delta\theta}{\Delta\theta} \cdot \frac{\Delta\theta}{\Delta t} + \Delta\theta\frac{\Delta\theta}{\Delta t}[\ ].$$

The quantity within the bracket [ ] is finite and remains finite for $\Delta\theta = 0$. Hence, passing to the limit for Δt equal to zero,

$$2A' = r^2\theta' = xy' - yx' = h. \tag{1}$$

The rate at which the radius vector sweeps over areas A' is called the *areal velocity.* The constant h is therefore twice the areal velocity.

It follows from Eq. (280.2) that if the force which is acting is a central force the areal velocity is constant; and conversely, if the areal velocity is constant,

$$\frac{1}{r}(r^2\theta')' = 0,$$

and, therefore, the force acting is a central force.

It will be seen from the geometry of the situation that the equations in Eq. (278.1) represent the projection of the areal velocity upon the three fundamental planes of reference, multiplied by the mass factor m.

284. The Differential Equation of the Orbit.—When the force which is acting is a central force, the time t can be eliminated altogether from the differential equation. In order to do this, let u be the reciprocal of r. Then the areas integral

$$r^2\theta' = h$$

becomes

$$\theta' = hu^2.$$

Also,

$$r' = \frac{dr}{d\theta}\cdot\theta' = -\frac{1}{u^2}\frac{du}{d\theta}\cdot\theta' = -h\frac{du}{d\theta};$$

$$r'' = -h\frac{d^2u}{d\theta^2}\theta' = -h^2u^2\frac{d^2u}{d\theta^2}.$$

On substituting these values of r'' and θ' in the first equation of Eq. (281.1), it becomes

$$h^2u^2\left(\frac{d^2u}{d\theta^2} + u\right) = -\frac{f}{m}, \tag{1}$$

which is the differential equation of the path of the particle. It must not be forgotten that f itself is positive for a repellant force and negative for an attractive force.

If f is a function of u alone, Eq. (1) can be integrated by multiplying through by the integrating factor $2/u^2 \cdot du/d\theta$. In this manner the integral

$$h^2\left[\left(\frac{du}{d\theta}\right)^2 + u^2\right] = -\frac{2}{m}\int\frac{f}{u^2}du.$$

is obtained. If p is the length of the perpendicular from the origin to the tangent,[1]

$$\frac{1}{p^2} = \left(\frac{du}{d\theta}\right)^2 + u^2.$$

Hence,

$$h^2\left[\left(\frac{du}{d\theta}\right)^2 + u^2\right] = \frac{h^2}{p^2} = s'^2 = -\frac{2}{m}\int f\frac{du}{u^2} = \frac{2}{m}\int f\cdot dr.$$

285. Given the Orbit to Determine the Force.—If the equation of the path is given with the center of force at the origin, u

[1] WILLIAMSON, "Differential Calculus," p. 224.

becomes a definite function of θ; and, since differentiation is always possible, Eq. (284.1) determines f as a function of u and θ, or, if θ is eliminated by means of the equation of the curve, as a function of u alone. Strictly speaking, the force is determined only along the path which is given, and along this path the elimination of θ is legitimate. If it is known that the force is everywhere a function of u alone, this elimination gives the desired expresssion for the force, but if no such restriction is given the force can be determined only on the given path.

II. THE HARMONIC LAW

286. The Orbit is an Ellipse with the Origin at the Center.—As an example of the determination of the law of force, which is assumed to be a function of the distance only, let the given orbit be an ellipse with the origin at its center.

The equation of the ellipse in polar coordinates is

$$r^2 = \frac{b^2}{1 - e^2 \cos^2 \theta},$$

where b is the minor axis and e is the eccentricity; hence,

$$bu = \sqrt{1 - e^2 \cos^2 \theta}.$$

On differentiating this expression twice with respect to θ, it is found that

$$b\frac{d^2u}{d\theta^2} = \frac{e^2 \cos^2 \theta - e^2 \sin^2 \theta}{(1 - e^2 \cos^2 \theta)^{\frac{1}{2}}} - \frac{e^4 \sin^2 \theta \cos^2 \theta}{(1 - e^2 \cos^2 \theta)^{\frac{3}{2}}};$$

and therefore,

$$\frac{d^2u}{d\theta^2} + u = \frac{1 - e^2}{b^4}\frac{1}{u^3}.$$

Equation (284.1) then gives

$$f = -mh^2u^2\left(\frac{d^2u}{d\theta^2} + u\right) = -\frac{mh^2}{a^2b^2}r.$$

The force is therefore directly proportional to the distance of the particle from the origin; and since the expression for f is negative, the force is an attractive one.

287. The Converse Problem.—The converse of the preceding problem is: If the force is attractive and proportional to the distance, what curves are possible orbits? According to Hooke's law for small strains (Sec. 208), the force of restitution is proportional to the displacement in many physical situations, such as tuning forks, vibrating strings, etc. If the factor of proportion-

ality is independent of the direction of the displacement, which is not always the case, the expression for the force is

$$f = -k^2mr.$$

For this law of force, Eq. (284.1) becomes

$$\frac{d^2u}{d\theta^2} + u = \frac{k^2}{h^2}\frac{1}{u^3}.$$

On multiplying through by $2du$, the first integral is found to be

$$\left(\frac{du}{d\theta}\right)^2 + u^2 = 2c - \frac{k^2}{h^2}\frac{1}{u^2}.$$

If this equation is multiplied through by u^2, and then for simplicity of notation the substitutions

$$w = u^2, \qquad \frac{k^2}{h^2} = g^2$$

are made, it becomes

$$\begin{aligned}\frac{1}{4}\left(\frac{dw}{d\theta}\right)^2 &= -g^2 + 2cw - w^2\\ &= (c^2 - g^2) - (w - c)^2.\end{aligned}$$

Therefore,

$$\frac{dw}{\sqrt{(c^2 - g^2) - (w - c)^2}} = \pm 2d\theta,$$

the integral of which is

$$\cos^{-1}\frac{w - c}{\sqrt{c^2 - g^2}} = 2(\theta - \theta_0 + \tfrac{1}{2}\pi);$$

so that

$$w = c - \sqrt{c^2 - g^2}\cos 2(\theta - \theta_0),$$

and therefore

$$r^2 = \frac{1}{(c + \sqrt{c^2 - g^2}) - 2\sqrt{c^2 - g^2}\cos^2(\theta - \theta_0)}.$$

On setting

$$c = \frac{2 - e^2}{2b^2}, \qquad g^2 = \frac{1 - e^2}{b^4},$$

the expression for r^2 becomes

$$r^2 = \frac{b^2}{1 - e^2\cos^2(\theta - \theta_0)},$$

which is the normal form of the equation of an ellipse with the origin at its center.

Since

$$b = a\sqrt{1 - e^2}, \qquad h^2 = \frac{k^2}{g^2},$$

it is found that

$$h = kab.$$

On substituting these values of r^2 and h in the areas integral

$$r^2\theta' = h$$

and then integrating, the time is obtained as a function of θ. This method, however, is very complicated for a matter that is otherwise very simple, as is shown in the next section.

288. The Same Problem in Rectangular Coordinates.—For most problems, the method of the preceding article is the simplest method, but it is not so for simple harmonic motion. On setting

$$f = -k^2mr$$

in Eq. (280.1), they become

$$\begin{aligned} x'' &= -k^2x, \\ y'' &= -k^2y, \end{aligned}$$

each of which is independent of the other, and represents a simple harmonic motion (Sec. 251). The solutions are, therefore,

$$\left.\begin{aligned} x &= A\cos kt + B\sin kt, \\ y &= C\cos kt + D\sin kt, \end{aligned}\right\} \tag{1}$$

where A, B, C, and D are the constants of integration. Solving these equations, it is found that

$$\cos kt = \frac{Dx - By}{AD - BC}, \qquad \sin kt = \frac{-Cx + Ay}{AD - BC}.$$

On squaring and then adding these two equations, the time is eliminated, with the result

$$(Dx - By)^2 + (Cx - Ay)^2 = (AD - BC)^2. \tag{2}$$

This is the equation of a central conic with the origin at the center, provided the right member is not zero. It is evident on dynamical grounds that the conic is an ellipse and not an hyperbola, since the force drawing the particle toward the origin increases with the distance, and in an hyperbola the distance increases indefinitely.

If the right member vanishes, the proportion

$$\frac{A}{C} = \frac{B}{D}$$

holds, provided C and D are not zero, and the equation reduces to the pair of coincident straight lines

$$\frac{D^2 + C^2}{C^2}(Cx - Ay)^2 = 0.$$

If

$$A = B = C = D = 0,$$

the solution is

$$x = 0, \qquad y = 0,$$

and the particle remains at rest at the origin.

The length of time necessary for one complete circuit of the ellipse, that is, the period, is

$$P = \frac{2\pi}{k} \tag{3}$$

which is independent of the constants of integration. Therefore the period does not depend upon the size of the ellipse.

289. The Hodograph.—By differentiation of Eq. (288.1), the following equations are obtained:

$$\begin{aligned} x' &= -Ak \sin kt + Bk \cos kt, \\ y' &= -Ck \sin kt + Dk \cos kt. \end{aligned}$$

The elimination of the time between these two equations gives the equation of the hodograph, namely,

$$(Dx' - By')^2 + (Cx' - Ay')^2 = k^2(AD - BC)^2. \tag{1}$$

A comparison of Eq. (1) with Eq. (288.2) shows that the hodograph is an ellipse similar to the path of the particle, but differing from it in size unless the value of k^2 is unity.

III. THE NEWTONIAN LAW WITH A FIXED CENTER

290. The Path is a Conic with the Origin at the Focus.—The equation of a conic with the origin at the focus is

$$r = \frac{p}{1 + e \cos \theta},$$

where

$$\begin{aligned} p &= a(1 - e^2) \qquad &&\text{for the ellipse,} \\ p &= a(e^2 - 1) &&\text{for the hyperbola,} \end{aligned}$$

a is the semiaxis major, and e is the eccentricity. Hence,

$$u = \frac{1}{p} + \frac{e}{p}\cos\theta.$$

rom this it follows that

$$\frac{d^2u}{d\theta^2} + u = \frac{1}{p};$$

therefore,

$$f = -h^2u^2\left(\frac{d^2u}{d\theta^2} + u\right) = -\frac{h^2}{p}\frac{m}{r^2}.$$

Consequently the force is attractive and its intensity is inversely proportional to the square of the distance from the origin.

291. Kepler's Laws of Planetary Motion.—The first two of Kepler's three laws of planetary motion were published in 1609, and the third in 1619. They are as follows:

I. The orbit of each planet is an ellipse with the sun at one focus.

II. The line which joins a planet to the sun sweeps over equal areas in equal intervals of time.

III. The squares of the periods of the planets are proportional to the cubes of their mean distances from the sun.

It was upon the three laws of Kepler and upon his own three laws of motion, that Newton founded the law of gravitation. It follows, for example, from Kepler's second law (Sec. 283) that the force which is acting upon each of the planets is always directed toward the sun, and is therefore a central force. It follows from his first law (Sec. 290) that the intensity of the force varies inversely as the square of the distance of the planet from the sun; and from the third law it follows (Sec. 309) that it is the same force which is acting upon each of the planets.

Suppose, for example, that Venus and Jupiter were made of iron and that the sun was a magnet, while the remainder of the planets were composed of non-magnetic material and were acted upon only by gravitation. Under these circumstances the first two laws of Kepler would hold for each of the planets, but the third would no longer hold. The fact that the third law does hold, however, proves that gravitation is the active agent in every case.

The merging of the three apparently independent laws of Kepler into the single and more fundamental law of gravitation by Newton is an interesting example of the process of unifying the activities of nature which is the goal of science. Another interesting example of the same kind is the inclusion of the domain of optics in the electromagnetic theory of James Clerk Maxwell.

292. The Law of Force is the Inverse Square.—Conversely let it be supposed, that the particle moves under an attractive central force which varies inversely as the square of the distance and then let the possible orbits be determined.

On setting

$$f = -mk^2u^2$$

in Eq. (284.1), it becomes

$$h^2u^2\left(\frac{d^2u}{d\theta^2} + u\right) = k^2u^2,$$

or

$$\frac{d^2u}{d\theta^2} + u = \frac{k^2}{h^2};$$

or again, if u is replaced by

$$u = \bar{u} + \frac{k^2}{h^2},$$

it becomes

$$\frac{d^2\bar{u}}{d\theta^2} + \bar{u} = 0,$$

the solution of which is (Sec. 251)

$$\bar{u} = A \cos(\theta - \theta_0).$$

Hence,

$$u = \frac{k^2}{h^2} + A \cos(\theta - \theta_0)$$

and

$$r = \frac{\dfrac{h^2}{k^2}}{1 + \dfrac{Ah^2}{k^2}\cos(\theta - \theta_0)}. \tag{1}$$

If this expression is compared with the equation of a conic

$$r = \frac{p}{1 + e\cos(\theta - \theta_0)},$$

it is seen that the orbit is a conic with the origin at one of the foci, and that

$$\frac{h^2}{k^2} = p, \qquad A\frac{h^2}{k^2} = Ap = e.$$

$$\left.\begin{array}{llll}\text{For an ellipse} & p = a(1 - e^2), & \therefore h = k\sqrt{a}\sqrt{1 - e^2}, & e < 1; \\ \text{for a parabola} & p = 2q, & \therefore h = k\sqrt{2q}, & e = 1; \\ \text{for a hyperbola} & p = a(e^2 - 1), & \therefore h = k\sqrt{a}\sqrt{e^2 - 1}, & e > 1.\end{array}\right\} \tag{2}$$

The letter q represents the distance from the focus to the vertex. For the sake of a name, the vertex nearest the center of force is called the *perihelion* point; therefore, q is the perihelion distance. In the case of an ellipse, the more remote vertex is called the *aphelion* point.

293. The Energy Equation.—The integral Eq. (281.3) is

$$\frac{1}{2}m(r'^2 + r^2\theta'^2) - U(r) = C,$$

where $U(r)$ is determined by the relation

$$\frac{dU}{dr} = f = -\frac{k^2m}{r^2};$$

therefore,

$$U = +\frac{k^2m}{r}.$$

Hence, the energy equation is

$$\frac{1}{2}m(r'^2 + r^2\theta'^2) - \frac{k^2m}{r} = C. \tag{1}$$

On eliminating θ' by means of the areas integral

$$r^2\theta' = h,$$

the energy integral becomes

$$\frac{1}{2}m\left(r'^2 + \frac{h^2}{r^2}\right) - \frac{k^2m}{r} = C. \tag{2}$$

The species of the conic is determined by the sign of the energy constant C. If C is negative, r cannot become infinite, for the terms which contain r in the denominator would vanish for r equal to infinity, and Eq. (2) would become

$$\frac{1}{2}mr'^2 = C,$$

which is impossible, for the left side of this expression is positive, or zero, while the right side is negative. Since the orbit is a conic, it must be an ellipse.

If C is zero, the particle has just sufficient speed to recede to infinity, that is, the limiting speed is equal to zero.

If C is positive, and Eq. (2) is written

$$\frac{1}{2}mr'^2 = \frac{k^2m}{r} - \frac{h^2m}{2r^2} + C, \tag{3}$$

it is seen that there is but one real, positive value of r, say $r = r_0$, for which the right member vanishes. The value of r cannot be less than r_0 since r'^2 cannot be negative, but it may have any larger value. If r' is negative, r decreases until the value r_0, is reached, after which r' is positive and r increases without limit. The orbit is therefore an hyperbola. The speed of the particle, however, has a finite limiting value as r increases.

Equation (3) holds for all points of the orbit, whether C is positive or negative; in particular, it is true at the perihelion point. At the perihelion point r' vanishes for all three types of conics, since the curve is perpendicular to the radius vector at that point. Also, at the perihelion point,

$$r = a(1 - e) \qquad \text{in the ellipse,}$$
$$r = q \qquad \text{in the parabola,}$$
and
$$r = a(e - 1) \qquad \text{in the hyperbola.}$$

If these values are substituted in Eq. (3), it is found that

$$C = \frac{m}{2}\frac{k^2a(1-e^2)}{a^2(1-e)^2} - \frac{k^2m}{a(1-e)} = -\frac{k^2m}{2a} \qquad \text{for the ellipse,}$$
$$C = \frac{m}{2}\frac{2k^2q}{q^2} - \frac{k^2m}{q} = 0 \qquad \text{for the parabola,}$$
$$C = \frac{m}{2}\frac{k^2a(e^2-1)}{a^2(e-1)^2} - \frac{k^2m}{a(e-1)} = +\frac{k^2m}{2a} \qquad \text{for the hyperbola.}$$

With these values of C, Eq. (1) becomes

$$\left.\begin{aligned} r'^2 + r^2\theta'^2 &= s'^2 = k^2\left(\frac{2}{r} - \frac{1}{a}\right) && \text{for the ellipse,}\\ r'^2 + r^2\theta'^2 &= s'^2 = k^2\left(\frac{2}{r} + 0\right) && \text{for the parabola,}\\ r'^2 + r^2\theta'^2 &= s'^2 = k^2\left(\frac{2}{r} + \frac{1}{a}\right) && \text{for the hyperbola.}\end{aligned}\right\} \quad (4)$$

294. The Radius Vector as a Function of a Parameter.—If the values of C and h^2 are substituted in Eq. (293.3) and then the factor $mk^2/2$ is removed, the following expressions result:

$$\left.\begin{aligned} \frac{r'^2}{k^2} &= -\frac{a(1-e^2)}{r^2} + \frac{2}{r} - \frac{1}{a} && \text{(ellipse),}\\ \frac{r'^2}{k^2} &= -\frac{2q}{r^2} + \frac{2}{r} && \text{(parabola),}\\ \frac{r'^2}{k^2} &= -\frac{a(e^2-1)}{r^2} + \frac{2}{r} + \frac{1}{a} && \text{(hyperbola).}\end{aligned}\right\} \quad (1)$$

These equations are readily put in the form:

$$\left.\begin{aligned}
\frac{rdr}{\sqrt{a^2e^2 - (a - r)^2}} &= \frac{kdt}{\sqrt{a}} && \text{(ellipse)},\\
\frac{rdr}{\sqrt{-4q^2 + 4rq}} &= \frac{kdt}{\sqrt{2q}} && \text{(parabola)},\\
\frac{rdr}{\sqrt{(a + r)^2 - a^2e^2}} &= \frac{kdt}{\sqrt{a}} && \text{(hyperbola)}.
\end{aligned}\right\} \tag{2}$$

Let new variables E, F, and G be introduced by the differential relations

$$\left.\begin{aligned}
dE &= \frac{k}{r}\frac{dt}{\sqrt{a}} && \text{(ellipse)},\\
dF &= \frac{k}{r}\frac{dt}{\sqrt{2q}} && \text{(parabola)},\\
dG &= \frac{k}{r}\frac{dt}{\sqrt{a}} && \text{(hyperbola)}.
\end{aligned}\right\} \tag{3}$$

Then equations (2) become

$$\left.\begin{aligned}
\frac{dr}{\sqrt{a^2e^2 - (a - r)^2}} &= dE && \text{(ellipse)},\\
\frac{dr}{\sqrt{-4q^2 + 4qr}} &= dF && \text{(parabola)},\\
\frac{dr}{\sqrt{(a + r)^2 - a^2e^2}} &= dG && \text{(hyperbola)}.
\end{aligned}\right\} \tag{4}$$

If the constants of integration are chosen so that E, F, and G vanish at the perihelion points, the integrals of Eq. (4) are

$$\left.\begin{aligned}
r &= a(1 - e\cos E) && \text{(ellipse)},\\
r &= q(1 + F^2) && \text{(parabola)},\\
r &= a(e\cosh G - 1) && \text{(hyperbola)}.
\end{aligned}\right\} \tag{5}$$

295. The Polar Angle as a Function of the Parameter.—If, in the areas integral

$$r^2\theta' = h,$$

the independent variable is changed by means of Eq. (294.3), and the values of h and r are substituted from Eq. (292.2) and (294.5), the following equations result:

$$\left.\begin{aligned}
d\theta &= \frac{\sqrt{1 - e^2}\,dE}{1 - e\cos E} && \text{(ellipse)},\\
d\theta &= \frac{2dF}{1 + F^2} && \text{(parabola)},\\
d\theta &= \frac{\sqrt{e^2 - 1}\,dG}{e\cosh G - 1} && \text{(hyperbola)}.
\end{aligned}\right\} \tag{1}$$

If θ is measured from the perihelion points, the integrals of these expressions are[1]

$$\left.\begin{aligned}
\tan\frac{\theta}{2} &= \sqrt{\frac{1+e}{1-e}}\tan\frac{E}{2} = \cot\frac{\varphi}{2}\tan\frac{E}{2}, && \text{(ellipse)},\\
\tan\frac{\theta}{2} &= F, && \text{(parabola)},\\
\tan\frac{\theta}{2} &= \sqrt{\frac{e+1}{e-1}}\tanh\frac{G}{2} = \coth\frac{\psi}{2}\tanh\frac{G}{2}, && \text{(hyperbola)}.
\end{aligned}\right\}\quad(2)$$

Since e is always less than unity in the ellipse, and greater than unity in the hyperbola, the substitutions

$$e = \cos\varphi \text{ (ellipse)} \qquad \text{and} \qquad e = \cosh\psi \text{ (hyperbola)},$$

are always possible.

296. The Time as a Function of the Parameter.—In order to complete the solution of the problem, it is necessary to express the time also as a function of the parameter. On substituting the value of r from Eq. (294.5) in Eq. 294.3, it is found by a very simple integration that

$$\left.\begin{aligned}
\frac{k}{a^{\frac{3}{2}}}(t-T) &= E - e\sin E && \text{(ellipse)},\\
\frac{k}{\sqrt{2q^{\frac{3}{2}}}}(t-T) &= F + \frac{1}{3}F^3 && \text{(parabola)},\\
\frac{k}{a^{\frac{3}{2}}}(t-T) &= e\sinh G - G && \text{(hyperbola)},
\end{aligned}\right\}\quad(1)$$

the constant of integration having been chosen so that t has the value T at the perihelion point.

297. The True, Mean, and Eccentric Anomalies.—If, for simplicity of notation, the substitution

$$M = \frac{k}{a^{\frac{3}{2}}}(t-T) \qquad (1)$$

is made when the motion is in an ellipse, the first equation of Eq. (296.1) becomes

$$M = E - e\sin E, \qquad (2)$$

and is known as *Kepler's equation*, although Kepler did not derive it in this manner. It is evident that M and E can be regarded as angles, and when they are so regarded they are called

[1] PEIRCE, "A Short Table of Integrals" (Numbers 300, 47, 471).

the *mean anomaly* and the *eccentric anomaly*, respectively. The polar angle θ when measured from the perihelion point is called the *true anomaly.*

The first equation of Eq. (295.2) shows that as the true anomaly increases from 0 to π the eccentric anomaly also increases from 0 to π, and that in this interval the true anomaly is always greater than the eccentric anomaly. As the true anomaly increases from π to 2π, the eccentric anomaly also increases from π to 2π, but in this second half of the circle the eccentric anomaly is greater than the true anomaly. Thus the true and the eccentric anomalies have the same value $2n\pi$ at the perihelion point; and the same value $(2n+1)\pi$ at the aphelion points. Kepler's equation (Eq. (2)) shows that in the first half-circle the mean anomaly is less than the eccentric anomaly, and is greater in the second half-circle. Thus all three anomalies have the same values at the perihelion and aphelion points, and

$$\theta > E > M \text{ in the first and second quadrants,}$$
$$\theta < E < M \text{ in the third and fourth quadrants.}$$

298. Derivation of Kepler's Equation Directly from Kepler's Laws.—Let C be the center and F the focus of an ellipse (Fig. 151); CA and CB the major and minor semiaxes a and b; and P the position of the particle in the ellipse. Draw the auxiliary circle, and let Q be the point where the perpendicular to the major axis PS cuts the auxiliary circle. Draw CP, CQ, FP, and FQ. The radius vector r is FP.

Since all of the ordinates of the ellipse are in the ratio of b/a to the corresponding ordinates of the auxiliary circle, the equation

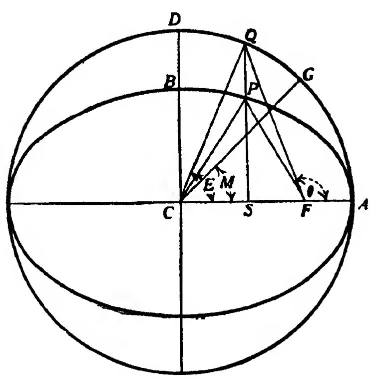


Fig. 151.

$$r^2 = \overline{PF}^2 = \overline{PS}^2 + \overline{SF}^2$$

becomes

$$\begin{aligned} r^2 &= b^2 \sin^2 E + (ae - a \cos E)^2 \\ &= a^2[1 - 2e \cos E + e^2 \cos^2 E], \end{aligned}$$

and therefore,

$$r = a(1 - e \cos E). \tag{1}$$

On comparing Eq. (1) with the first equation of Eq. (294.5), it is evident that the angle ACQ is the eccentric anomaly. The angle AFP is the true anomaly.

Imagine a fictitious particle G which starts at A at the same instant as P, moves in the circle with uniform angular speed with respect to the center, and arrives back at A at the same instant that P does; P, however, having traveled in the ellipse in accordance with Kepler's laws. The angle ACG is then the mean anomaly M. Since the angle M and the sectorial area AFP of the ellipse are each proportional to the time

$$\frac{M}{2\pi} = \frac{\text{sector } AFP}{\text{area of ellipse}} = \frac{\text{sector of circle } AFQ}{\text{area of circle}}.$$

But

$$\begin{aligned}\text{sector } AFQ &= \text{sector } ACQ - \text{triangle } FCQ \\ &= \frac{1}{2}a^2E - \frac{1}{2}a^2e\sin E.\end{aligned}$$

Hence,

$$\frac{M}{2\pi} = \frac{a^2}{2}\,\frac{E - e\sin E}{\pi a^2},$$

and therefore,

$$M = E - e\sin E,$$

which is Kepler's equation (Eq. (297.2)).

If a circle of radius ae, which is equal to CF, be drawn with C as center, it will cut the radius CQ in a point Q_1 (not shown in the diagram). The perpendicular Q_1S_1 to the major axis CA is equal to $ae \sin E$. If the arc QG is taken equal to the straight line Q_1S_1, then the angle ACG is the mean anomaly M.

299. Analogous Discussion for the Hyperbola.—The parametric equations of the equilateral hyperbola Q_1AQ (Fig. 152) are

$$\begin{aligned}x &= -a\cosh G, \\ y &= +a\sinh G,\end{aligned}$$

the center of the hyperbola being at the origin of coordinates C. It will be observed that for positive values of y the parameter G is positive, and for negative values of y it is negative. On eliminating G between these two equations, the equation of the hyperbola in its normal form is obtained, namely,

$$x^2 - y^2 = a^2.$$

The differential of the area swept over by the radius vector is

$$\begin{aligned}2dA &= xdy - ydx, \\ &= -a^2dG.\end{aligned}$$

Hence, if G and A vanish at the vertex of the hyperbola,

$$2A = -a^2G.$$

That is, the parameter G is, apart from sign, twice the area CAQ, shaded in the diagram, divided by a^2, where a is the distance AC.

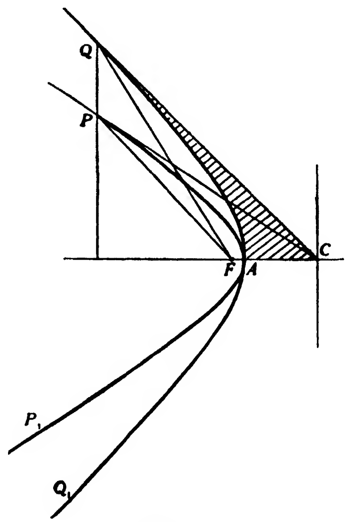


FIG. 152.

Consider now any hyperbola P_1AP which has the same vertex A and the same axis AC. It has the same major axis a as the equilateral hyperbola, but its minor axis is

$$b = a\sqrt{e^2 - 1}, \qquad e > 1.$$

The parametric equations of this hyperbola are

$$\left.\begin{aligned} x &= -a \cosh G, \\ y &= +b \sinh G, \end{aligned}\right\} \tag{1}$$

where G is the same parameter that was used for the equilateral hyperbola. It is evident that for the same value of x, and therefore for the same value of G also, the ordinates of the hyperbola P_1AP are in the ratio

$$\frac{b}{a} = \sqrt{e^2 - 1}$$

to the ordinates of the equilateral hyperbola Q_1AQ.

Let F be the focus of the hyperbola P_1AP. The rate at which the radius vector FP is sweeping over areas is, by Eq. (292.2),

$$2A' = k\sqrt{a}\sqrt{e^2 - 1},$$

and therefore, if A is measured from the perihelion,

$$A = \frac{1}{2}k\sqrt{a}\sqrt{e^2 - 1}(t - T);$$

or, if

$$N = \frac{k(t - T)}{a^{\frac{3}{2}}}, \tag{2}$$

$$\text{sector } FAP = \frac{1}{2}a^2\sqrt{e^2 - 1}N.$$

Also

$$\text{sector } FAP = \sqrt{e^2 - 1} \times \text{sector } FAQ$$

and $$\text{sector } FAQ = \text{triangle } FCQ - \text{area } ACQ,$$

both the area of the triangle and the shaded area being regarded as positive. Therefore,

$$\text{sector } FAQ = \frac{1}{2}a^2e \sinh G - \frac{1}{2}a^2G$$

and

$$\frac{1}{2}a^2\sqrt{e^2 - 1}\,N = \sqrt{e^2 - 1}\left[\frac{1}{2}a^2e \sinh G - \frac{1}{2}a^2G\right].$$

On removing the superfluous factor $a^2\sqrt{e^2 - 1}/2$ there remains

$$N = e \sinh G - G, \tag{3}$$

which is the third equation of Eq. (296.1) and the analogue of Kepler's equation.

It is seen from Eq. (293.4) that the limiting value of the speed of the particle in hyperbolic motion, that is, for r infinite, is

$$s' = \frac{k}{\sqrt{a}}.$$

If a fictitious particle p starts from C when the real particle P is at A, and moves along the asymptote to the equilateral hyperbola Q_1AQ with a constant speed equal to the limiting speed of the real particle, then

$$N = \frac{\text{twice the area of the triangle } F_0Cp}{a^2};$$

for the base of this triangle is

$$Cp = \frac{k}{\sqrt{a}}(t - T),$$

and its altitude is the perpendicular from F to the asymptote, which is a. (F_0 = focus of equilateral hyperbola.)

The area of the triangle is, therefore, $k\sqrt{a}(t - T)/2$, and

$$\frac{\text{twice the area of triangle } FCp}{a^2} = \frac{k}{a^{\frac{3}{2}}}(t - T) = N.$$

300. The Period of Elliptic Motion.—From the areas integral and the first equation of Eq. (292.2) it is found that for elliptic motion

$$2A' = h = k\sqrt{a}\sqrt{1 - e^2},$$

which, after integration, becomes

$$2A = k\sqrt{a}\sqrt{1 - e^2}(t - T).$$

When the radius vector has swept over the entire ellipse, A is equal to πab and the time $t - T$ is equal to the period P of the motion. Hence,

$$2\pi ab = k\sqrt{a}\sqrt{1 - e^2}\,P$$

which reduces to

$$P = \frac{2\pi a^{\frac{3}{2}}}{k}. \tag{1}$$

The same result is obtained immediately from the expression for the mean anomaly (Eq. (297.1)). It will be observed that the period depends only upon the major axis of the ellipse, and not at all upon the eccentricity, a result that one would scarcely anticipate.

For particles at different distances from the center of force, Eq. (1) leads to the proportion

$$P_1^2 : P_2^2 :: a_1^3 : a_2^3,$$

which is Kepler's third law.

301. The Hodograph for the Inverse Square Law.—Setting

$$f = -\frac{k^2 m}{r^2}$$

in Eq. (280.1) the differential equations in rectangular coordinates become

$$x'' = -k^2\frac{x}{r^3}, \qquad y'' = -k^2\frac{y}{r^3}; \tag{1}$$

and the areas integral is

$$h = xy' - yx'. \tag{2}$$

If the left member of the first equation of Eq. (1) is multiplied by the left member of Eq. (2), and the right member of Eq. (1) by the right member of Eq. (2), there results

$$\begin{aligned} hx'' &= -k^2\frac{x^2y' - xyx'}{r^3} = -k^2\frac{(x^2 + y^2)y' - y(xx' + yy')}{r^3} \\ &= -k^2\frac{r^2y' - yrr'}{r^3} = -k^2\left(\frac{y}{r}\right)'. \end{aligned}$$

Therefore, by integration,

$$hx' = -k^2\frac{y}{r}; \tag{3}$$

the constant of integration being zero if the major axis of the conic coincides with the x-axis, since x' then vanishes when y vanishes.

On multiplying together the second equation of Eq. (1) and the areas integral (Eq. (2)), it is found that

$$hy'' = -k^2\frac{xyy' - y^2x'}{r^3} = -k^2\frac{x(yy' + xx') - (x^2 + y^2)x'}{r^3},$$

$$= -k^2\frac{rxr' - r^2x'}{r^3} = +k^2\left(\frac{x}{r}\right)'.$$

Therefore, by integration,

$$hy' = +k^2\left(\frac{x}{r}\right) + \text{const.} \tag{4}$$

At the perihelion point

$$x = q, \qquad y = 0,$$

$$x' = 0, \qquad y' = \frac{h}{q} \qquad \text{(from Eq. (2))}.$$

If these values are substituted in Eq. (4), it is found that the constant of integration is

$$\text{const} = \frac{h^2}{q} - k^2 = k^2\left(\frac{p}{q} - 1\right) = k^2e$$

for all three species of conics, q being the perihelion distance and p the parameter (see Eq. (292.2)). Therefore,

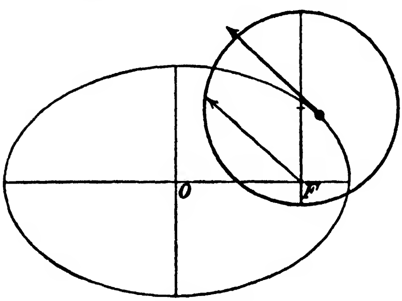


Fig. 153.

$$hy' = +k^2\left(\frac{x}{r} + e\right). \tag{5}$$

Equations (3) and (5) can be written

$$\left.\begin{aligned} x' &= -\frac{k}{\sqrt{p}}\frac{y}{r}, \\ \left(y' - \frac{ke}{\sqrt{p}}\right) &= \frac{k}{\sqrt{p}}\frac{x}{r}; \end{aligned}\right\} \tag{6}$$

therefore,

$$x'^2 + \left(y' - \frac{ke}{\sqrt{p}}\right)^2 = \frac{k^2}{p}, \tag{7}$$

which is the equation of the hodograph. Equation (7) shows that the hodograph is a circle whatever the value of the eccentricity may be. The radius of the circle is $k/\sqrt{p}$, and its center is on the latus rectum through the origin at a distance $ke/\sqrt{p}$ from the origin. The origin, therefore, lies inside of the circle if the orbit is an ellipse, on the circle if it is a parabola, and outside of the circle if the orbit is an hyperbola.

302. Expansion of the Eccentric Anomaly in Series.—If z is defined as a function of w by the equation

$$z = w + \alpha\varphi(z), \tag{1}$$

where α is a parameter which is sufficiently small, it is possible to expand z as a power series in α, the coefficients of which are functions of w, by means of the very beautiful formula of Lagrange[1]

$$\left.\begin{aligned} z = w + \alpha\varphi(w) &+ \frac{\alpha^2}{2!}\frac{d}{dw}(\varphi(w))^2 + \cdots \\ &+ \frac{\alpha^n}{n!}\frac{d^{n-1}}{dw^{n-1}}(\varphi(w))^n + \cdots . \end{aligned}\right. \tag{2}$$

If Kepler's equation (Eq. (297.2)) is written

$$E = M + e \sin E,$$

it has the form of Eq. (1) for which Lagrange's equation (Eq. (2)) is applicable. On identifying E with z, M with w, and sin with φ, it is found that

$$\left.\begin{aligned} E = M &+ e \sin M + \frac{e^2}{2!}\sin 2M \\ &+ \frac{e^3}{3!2^2}(3^2 \sin 3M - 3 \sin M) \\ &+ \frac{e^4}{4!2^3}(4^3 \sin 4M - 4 \cdot 2^3 \sin 2M) \\ &+ \frac{e^5}{5!2^4}(5^4 \sin 5M - 5 \cdot 3^4 \sin 3M + 5 \cdot 2 \sin M) \\ &+ \frac{e^6}{6!2^5}(6^5 \sin 6M - 6 \cdot 4^5 \sin 4M + 6 \cdot 5 \cdot 4^2 \sin 2M) \\ &+ \cdots\cdots\cdots\cdots \end{aligned}\right\} \tag{3}$$

Laplace proved that this expansion is convergent for all values of M if the eccentricity e is less than $0.662743 \cdots$.

Certain properties of this expansion are easily proved from Kepler's equation. E, for example, is an odd function of M; therefore, Eq. (3) contains only sines of multiples of M. There are no cosines and no constant terms. Aside from the first term M there are no terms which are not periodic with the period 2π. The coefficients of odd powers of e contain only sines of odd multiples of M, and the coefficients of even powers of e contain only sines of even multiples of M. The proof of these properties will be left to the student.

[1] GOURSAT-HEDRICK, "Mathematical Analysis," vol. I, p. 404. WILLIAMSON, "Differential Calculus," p. 151.

303. The Integral $\int(E - M)dM$.—Let the first term of the right member of Eq. (302.3) be taken to the left side, and then let the equation be multiplied by $-dM$ and integrated term by term, the constant of integration being taken equal to zero. Let this integral be denoted by

$$I = -\int(E - M)dM.$$

Then

$$\left.\begin{aligned} I = &+ e \cos M + \frac{e^2}{2!2} \cos 2M \\ &+ \frac{e^3}{3!2^2}(3 \cos 3M - 3 \cos M) \\ &+ \frac{e^4}{4!2^3}(4^2 \cos 4M - 4.2^2 \cos 2M) \\ &+ \cdots\cdots\cdots\cdots ; \end{aligned}\right\} \tag{1}$$

and, since the right member contains only trigonometric terms, it is evident that

$$\int_0^{2\pi} IdM = 0. \tag{2}$$

From Kepler's equation, it is found that

$$\begin{aligned} -\int(E - M)dM &= -e\int \sin E\,dM \\ &= -e\int \sin E(1 - e \cos E)dE, \end{aligned}$$

since

$$dM = (1 - e \cos E)dE.$$

Hence,

$$I = c + e \cos E - \frac{1}{4}e^2 \cos 2E, \tag{3}$$

where c is some constant. In order to find the value of the constant c, multiply Eq. (3) by dM and integrate from zero to 2π. The left member vanishes, by Eq. (2). Hence,

$$\begin{aligned} 0 &= \int_0^{2\pi}\left[c + e \cos E - \frac{1}{4}e^2 \cos 2E\right]dM, \\ &= \int_0^{2\pi}\left[c + e \cos E - \frac{1}{4}e^2 \cos 2E\right](1 - e \cos E)dE; \end{aligned}$$

and therefore

$$c = \frac{1}{2}e^2.$$

With this value of c, Eq. (3) becomes

$$I = e \cos E + e^2\left[\frac{1}{2} - \frac{1}{4} \cos 2E\right]. \tag{4}$$

304. Expansions of Other Functions.—From the expansions for E (Eq. (302.3)) and I (Eq. (303.1)) for which the general terms can be written down, a variety of other useful expansions are very readily obtained. From Kepler's equation, for example, it follows that

$$\sin E = \frac{E - M}{e},$$

and, therefore, by Eq. (302.3),

$$\sin E = \sin M + \frac{e}{2}\sin 2M + \frac{e^2}{8}(3 \sin 3M - \sin M) + \cdots \quad (1)$$

The eccentric anomaly is a function of the two independent variables e and M. On bearing this in mind, it is found from Kepler's equation,

$$M = E - e \sin E,$$

that

$$\frac{dE}{dM} = \frac{1}{1 - e\cos E}, \qquad \frac{dE}{de} = \frac{\sin E}{1 - e \cos E}. \quad (2)$$

Likewise, from Eq. (303.4)

$$\frac{dI}{dM} = -e \sin E, \qquad \frac{dI}{de} = \cos E + \frac{1}{2}e. \quad (3)$$

It must be remembered, in differentiating I with respect to e, that I is a function of e not only explicitly, but also implicitly through E.

From the second of Eq. (3) the following is obtained:

$$\cos E = \frac{dI}{de} - \frac{1}{2}e. \quad (4)$$

Therefore, by Eq. (303.1),

$$\cos E = \cos M + \frac{e}{2}(\cos 2M - 1) + \frac{3}{8}e^2(\cos 3M - \cos M) + \cdots \quad (5)$$

Since, by Eq. (294.5),

$$\frac{r}{a} = 1 - e \cos E = 1 + \frac{1}{2}e^2 - e\frac{dI}{de},$$

it follows at once that

$$\frac{r}{a} = 1 - e \cos M + \frac{e^2}{2}(1 - \cos 2M) + \frac{3}{8}e^3(\cos M - \cos 3M) + \cdots \quad (6)$$

Similarly,

$$\left(\frac{r}{a}\right)^2 = (1 - e\cos E)^2 = \left(1 + \frac{1}{2}e^2\right) - 2e\cos E + \frac{1}{2}e^2\cos 2E.$$

On comparing this expression with Eq. (303.4), it is seen that

$$\left(\frac{r}{a}\right)^2 = 1 + \frac{3}{2}e^2 - 2I.$$

Hence,

$$\left(\frac{r}{a}\right)^2 = 1 - 2e\cos M + e^2\left(\frac{3}{2} - \frac{1}{2}\cos 2M\right) + e^3\left(\frac{3}{8}\cos M - \frac{3}{8}\cos 3M\right) + \cdots \quad (7)$$

From the first of Eq. (2) the following equation is obtained:

$$\frac{a}{r} = \frac{1}{1 - e\cos E} = \frac{dE}{dM}.$$

Hence, from Eq. (302.3),

$$\frac{a}{r} = 1 + e\cos M + e^2\cos 2M + e^3\left(\frac{9}{8}\cos 3M - \frac{1}{8}\cos M\right) + \cdots \quad (8)$$

From the relations

$$r = a(1 - e\cos E) = \frac{a(1 - e^2)}{1 + e\cos\theta},$$

the following equations are obtained:

$$\left.\begin{aligned} \cos\theta &= \frac{\cos E - e}{1 - e\cos E} = -\frac{d}{de}\left(\frac{r}{a}\right), \\ \sin\theta &= \frac{\sqrt{1 - e^2}\sin E}{1 - e\cos E} = \sqrt{1 - e^2}\frac{dE}{de}. \end{aligned}\right\} \quad (9)$$

Therefore,

$$\cos\theta = \cos M + e(\cos 2M - 1) + \frac{9}{8}e^2(\cos 3M - \cos M) + \cdots. \quad (10)$$

$$\sin\theta = \sqrt{1 - e^2}\left[\sin M + e\sin 2M + \frac{3e^2}{8}(3\sin 3M - \sin M) + \cdots\right]. \quad (10)$$

305. The Force is the Inverse Square, but Repellant.—If the force varies inversely as the square of the distance, but is

repellant instead of attractive, it is necessary merely to change k^2 into $-k^2$ in Sec. 292. The steps in the process of integration are just the same, but there are certain changes in the signs of the constants of integration which must be made. Thus it is seen from Eq. (292.1) that the path of the particle is

$$r = \frac{p}{e \cos(\theta - \theta_0) - 1}, \tag{1}$$

which is that branch of the hyperbola which is convex toward the origin. On comparing this with the branch of the hyperbola which is concave toward the origin, namely,

$$r = \frac{p}{e \cos(\theta - \theta_0) + 1},$$

it is seen that the effect of changing k^2 into $-k^2$ is to change e into $-e$ and p into $-p$ (or a into $-a$).

The energy integral (Eq. (293.2)) becomes

$$\frac{1}{2}m(r'^2 + r^2\theta'^2) + \frac{k^2 m}{r} = C; \tag{2}$$

but since the perihelion is now $a(1 + e)$, instead of $a(e - 1)$ as before, the value of C is found to be

$$C = +\frac{k^2 m}{2a}, \tag{3}$$

which is the same as for the hyperbolic case in Sec. 293. This could have been anticipated; for, changing k^2 into $-k^2$ and a into $-a$ in Eq. (3) leaves the expression for C unaltered.

Similarly,

$$\left.\begin{aligned} r^2\theta' &= h = \sqrt{k^2 a(e^2 - 1)}, \\ \frac{r'^2}{k^2} &= -\frac{a(e^2 - 1)}{r^2} - \frac{2}{r} + \frac{1}{a}, \\ s'^2 &= k^2\left(-\frac{2}{r} + \frac{1}{a}\right); \end{aligned}\right\} \tag{4}$$

therefore, the particle arrives at infinity with a finite speed.

Changes occur in the following equations, which are numbered the same as the corresponding equations already given:

$$\frac{dr}{\sqrt{(r - a)^2 - a^2e^2}} = dG = \frac{k\,dt}{r\sqrt{a}}, \tag{294.4'}$$

$$r = a(1 + e \cosh G), \tag{294.5'}$$

$$d\theta = \frac{\sqrt{e^2 - 1}\, dG}{1 + e \cosh G}, \tag{295.1'}$$

$$\frac{k(t - T)}{a^{\frac{3}{2}}} = G + e \sinh G, \tag{296.1'}$$

$$\tan \frac{1}{2}\theta = \sqrt{\frac{e - 1}{e + 1}} \tanh \frac{1}{2}G, \tag{295.2'}$$

and

$$x'^2 + \left(y' - \frac{ke}{\sqrt{p}}\right)^2 = \frac{k^2}{p}. \tag{301.7'}$$

306. Real Orbits in Imaginary Time.—The constant k^2 which occurs in the expression for the force in the right members of the differential equations in Eq. (277.1) can be associated with t^2, since

$$x'' = k^2 \frac{d^2x}{d(kt)^2},$$

and the factor k^2, which is common to the right and the left members of the differential equations, can then be removed. If the force is attractive, $-k^2$ occurs; if repellant, $+k^2$ occurs. The change that occurs in the differential equations when k^2 is replaced by $-k^2$ is exactly the same that would occur if t were changed into it ($i = \sqrt{-1}$). Hence, the curious analytical result that real orbits which are described under a repellant force are still real orbits under an attractive force, but the time in which they are described is purely imaginary.

Similarly, real orbits in real time under an attractive force are real orbits in purely imaginary time under a repellant force. The hyperbolic motion, therefore, which occurs under a repellant inverse-square force is, from an analytic point of view, a periodic motion with the purely imaginary period (Eq. (300.1))

$$P = \frac{2\pi i a^{\frac{3}{2}}}{k}.$$

IV. THE TWO-BODY PROBLEM

307. The Two-body Problem.—Two bodies which attract each other as though they were particles are tossed out into empty space at random, and thereafter are subject to no force except their own mutual gravitational attraction. It is assumed that

the initial positions and velocities of the particles are given, and it is required to find their positions and velocities at any specified time in the future. This is the famous two-body problem which was solved by Newton.

Let the two bodies be m_1 and m_2 (Fig. 154) with the coordinates x_1, y_1, z_1 and x_2, y_2, z_2, respectively. According to the third law of motion (Sec. 45), the force which m_1 exerts on m_2 is equal and opposite to the force which m_2 exerts on m_1. In accordance with the law of gravitation this common attraction is along the line which joins them and in magnitude is

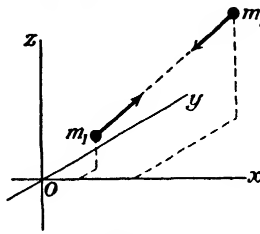


FIG. 154.

$$f = \kappa^2 \frac{m_1 m_2}{r^2_{12}},$$

where r_{12} is the distance between them. Resolving these forces into their components along the x-, y-, and z-axes, it is found that the differential equations are

$$\left.\begin{aligned}
m_1 x_1'' &= -\kappa^2 \frac{m_1 m_2}{r^2_{12}} \cdot \frac{x_1 - x_2}{r_{12}} = -\kappa^2 m_1 m_2 \frac{x_1 - x_2}{r^3_{12}},\\
m_1 y_1'' &= -\kappa^2 \frac{m_1 m_2}{r^2_{12}} \cdot \frac{y_1 - y_2}{r_{12}} = -\kappa^2 m_1 m_2 \frac{y_1 - y_2}{r^3_{12}},\\
m_1 z_1'' &= -\kappa^2 \frac{m_1 m_2}{r^2_{12}} \cdot \frac{z_1 - z_2}{r_{12}} = -\kappa^2 m_1 m_2 \frac{z_1 - z_2}{r^3_{12}};
\end{aligned}\right\} \quad (1)$$

$$\left.\begin{aligned}
m_2 x_2'' &= -\kappa^2 \frac{m_1 m_2}{r^2_{12}} \cdot \frac{x_2 - x_1}{r_{12}} = -\kappa^2 m_1 m_2 \frac{x_2 - x_1}{r^3_{12}},\\
m_2 y_2'' &= -\kappa^2 \frac{m_1 m_2}{r^2_{12}} \cdot \frac{y_2 - y_1}{r_{12}} = -\kappa^2 m_1 m_2 \frac{y_2 - y_1}{r^3_{12}},\\
m_2 z_2'' &= -\kappa^2 \frac{m_1 m_2}{r^2_{12}} \cdot \frac{z_2 - z_1}{r_{12}} = -\kappa^2 m_1 m_2 \frac{z_2 - z_1}{r^3_{12}};
\end{aligned}\right\} \quad (2)$$

where

$$r_{12} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

It will be observed that the differential equations are of the twelfth order, and therefore twelve integrals are necessary for the complete solution of the problem. It is sometimes convenient to think of the twelve constants of integration as corresponding to the six initial coordinates of position and the six initial coordinates of velocity, but other interpretations are possible, and, indeed, usually desirable.

If the first equations of Eqs. (1) and (2) are added together, and similarly, the second equations, and the third equations, the following equations result:

$$\left.\begin{aligned} m_1x_1'' + m_2x_2'' &= 0, \\ m_1y_1'' + m_2y_2'' &= 0, \\ m_1z_1'' + m_2z_2'' &= 0. \end{aligned}\right\} \tag{3}$$

Let $\bar{x}$, $\bar{y}$, and $\bar{z}$ be the coordinates of the center of gravity of the two bodies, and let M_{12} be the sum of the two masses m_1 and m_2. Then

$$\left.\begin{aligned} M_{12}\bar{x} &= m_1x_1 + m_2x_2, \\ M_{12}\bar{y} &= m_1y_1 + m_2y_2, \\ M_{12}\bar{z} &= m_1z_1 + m_2z_2, \end{aligned}\right\} \tag{4}$$

$$M_{12} = m_1 + m_2;$$

and Eqs. (3) become

$$M_{12}\bar{x}'' = 0, \qquad M_{12}\bar{y}'' = 0, \qquad M_{12}\bar{z}'' = 0.$$

The first integrals of these three equations are

$$\left.\begin{aligned} M_{12}\bar{x}' &= M_{12}s \cos\alpha, \\ M_{12}\bar{y}' &= M_{12}s \cos\beta, \\ M_{12}\bar{z}' &= M_{12}s \cos\gamma, \end{aligned}\right\} \tag{5}$$

where s, α, β, γ are constants and

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1. \tag{6}$$

The equations in Eq. (5) are the three components of the total momentum of the system. Since each component separately is constant, it follows that *the momentum of the system is constant.* On removing the factor M_{12} from each equation, then squaring and adding, there results

$$\bar{x}'^2 + \bar{y}'^2 + \bar{z}'^2 = s^2. \tag{7}$$

Therefore, *the center of gravity of the system moves with the constant speed s.*

Omitting the factor M_{12}, the integration of the equations in Eq. (5) gives

$$\left.\begin{aligned} \bar{x} &= s \cos\alpha \cdot t + \bar{x}_0, \\ \bar{y} &= s \cos\beta \cdot t + \bar{y}_0, \\ \bar{z} &= s \cos\gamma \cdot t + \bar{z}_0, \end{aligned}\right\} \tag{8}$$

where $\bar{x}_0$, $\bar{y}_0$, and $\bar{z}_0$ are the three constants of integration. The equations in Eq. (8), however, are the parametric equations of a straight line which passes through the point $\bar{x}_0$, $\bar{y}_0$, $\bar{z}_0$ and has α,

β, and γ as its direction angles. Hence, *the center of gravity of the system moves along a straight line with constant speed.*

The six integrals in Eqs. (5) and (8) are known as *the center of gravity integrals.*

308. The Relative Motion.—Let x, y, and z be the coordinates of m_2 with respect to m_1. Then

$$x_2 - x_1 = x, \qquad y_2 - y_1 = y, \qquad z_2 - z_1 = z; \tag{1}$$

also,

$$\begin{aligned} m_2x_2 + m_1x_1 &= 0, \\ m_2y_2 + m_1y_1 &= 0, \\ m_2z_2 + m_1z_1 &= 0. \end{aligned}$$

if the origin is taken at the center of gravity of the two bodies. From these two sets of equations it is found that

$$\left.\begin{aligned} x_1 &= \frac{-m_2x}{m_1 + m_2}, & x_2 &= \frac{m_1x}{m_1 + m_2}, \\ y_1 &= \frac{-m_2y}{m_1 + m_2}, & y_2 &= \frac{m_1y}{m_1 + m_2}, \\ z_1 &= \frac{-m_2z}{m_1 + m_2}, & z_2 &= \frac{m_1z}{m_1 + m_2}. \end{aligned}\right\} \tag{2}$$

The substitution of these values of $x_1, y_1, z_1; x_2, y_2, z_2$ in Eqs. (307.1) and (307.2) reduces the differential equations to the single set

$$\left.\begin{aligned} x'' &= -\kappa^2(m_1 + m_2)\frac{x}{r^3}, \\ y'' &= -\kappa^2(m_1 + m_2)\frac{y}{r^3}, \\ z'' &= -\kappa^2(m_1 + m_2)\frac{z}{r^3}. \end{aligned}\right\} \tag{3}$$

$$r^2 = x^2 + y^2 + z^2.$$

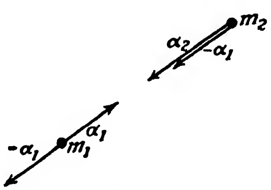


FIG. 155.

The equations in Eq. (3) have been derived analytically, but they can also be derived by an appeal to intuition. The forces which act upon the two bodies are equal and opposite, but the accelerations, although they are oppositely directed, are not equal. Indeed

$$m_1\alpha_1 + m_2\alpha_2 = 0.$$

If equal accelerations $-\alpha_1$ are applied to both bodies (Fig. 155) their *relative* motion will not be altered, and m_1 will be without acceleration. The acceleration of m_2 is then

$$\alpha_2 - \alpha_1 = \alpha_2 + \frac{m_2}{m_1}\alpha_2 = \frac{m_1 + m_2}{m_1}\alpha_2.$$

The vector α_2 is directed toward m_1 and in magnitude is equal to

$$\alpha_2 = \frac{\kappa^2 m_1}{r^2}.$$

Hence, the acceleration of m_2 with respect to m_1 is directed toward m_1 and in magnitude is equal to

$$\frac{\kappa^2(m_1 + m_2)}{r^2};$$

from which Eq. (3) follows at once. They are the same as the equations of motion of a particle about a *fixed center*, for which (Eq. (277.1))

$$f = -\kappa^2 \frac{m(m_1 + m_2)}{r^2}.$$

309. Kepler's Third Law.—Comparing this expression for the force with that which was used in Sec. 292, it is seen that all of the results which were obtained in the discussion for the fixed center (Secs. 292 to 306) are applicable, provided the k^2 which is used in that discussion is replaced by $\kappa^2(m_1 + m_2)$, where κ^2 is the gravitational constant. Therefore, the motion of m_2 with respect to m_1 is a conic with its focus at m_1.

If the orbit is an ellipse the motion is periodic with the period (Eq. (300.1))

$$P = \frac{2\pi a^{\frac{3}{2}}}{\kappa\sqrt{m_1 + m_2}}. \tag{1}$$

Consider the motion of two planets about the sun. Let S be the mass of the sun and m_1 and m_2 be the masses of two of the planets. Then, as in Sec. 300,

$$\frac{P_1^2}{P_2^2} = \frac{\kappa_2^2(S + m_2)}{\kappa_1^2(S + m_1)} \cdot \frac{a_1^3}{a_2^3}.$$

According to Kepler's third law, however, the squares of the periods of the planets are proportional to the cubes of their mean distances. Hence, so far as Kepler could detect, the factor

$$\frac{\kappa_2^2(S + m_2)}{\kappa_1^2(S + m_1)}$$

is equal to unity. Actually m_1 and m_2 are not equal, but in the solar system they are very small as compared with S, so that its largest possible value is

$$\frac{S + m_2}{S + m_1} = \frac{1048}{1047},$$

the mass of Jupiter being 1/1047 of the mass of the sun. Neglecting this slight discrepancy, Kepler's third law states that κ_1^2 is equal to κ_2^2, and therefore it is gravitation which is acting upon all of the planets, and not gravitation for some planets and magnetism or electrical attraction for others; for, in this last event κ_1^2 would not be equal to κ_2^2.

If the mean solar day is the unit of time, the mean distance of the earth from the sun is the unit of distance, and the mass of the sun is the unit of mass then in Eq. (1)

$$m_1 = 1, \qquad m_2 = .000003, \qquad P = 365.2564,$$

and therefore

$$\kappa = 0.0172021, \qquad \kappa^2 = 0.000295912,$$

and $$\log \kappa = 8.23558144-10, \qquad \log \kappa^2 = 6.471163 - 10$$

The constant κ, expressed in these units, is known as the *Gaussian Constant.*

V. THE INVERSE FIFTH POWER

310. The Force Varies Inversely as the Fifth Power of the Distance.—If the force of attraction is proportional to the inverse fifth power of the distance, that is, if

$$f = -k^2mu^5 = -\frac{k^2m}{r^5},$$

Equation (284.1) which is the differential equation of the path becomes

$$\frac{d^2u}{d\theta^2} + u = \frac{k^2}{h^2}u^3; \tag{1}$$

and the integral is

$$\left(\frac{du}{d\theta}\right)^2 + u^2 = \frac{k^2}{2h^2}u^4 + C, \tag{2}$$

where C is the constant of integration.

Let u_0 be the value of u and αu_0 the value of the derivative of u with respect to θ, when θ is equal to zero. Then

$$C = u_0^2\left(1 + \alpha^2 - \frac{k^2}{2h^2}u_0^2\right),$$

and Eq. (2) can be written

$$\left(\frac{d\frac{u}{u_0}}{d\theta}\right)^2 = \frac{k^2u_0^2}{2h^2}\left(\frac{u}{u_0}\right)^4 - \left(\frac{u}{u_0}\right)^2 + \left(1 + \alpha^2 - \frac{k^2u_0^2}{2h^2}\right);$$

or, putting

$$\frac{k^2u_0^2}{2h^2} = \frac{1}{2}b, \qquad \left(1 + \alpha^2 - \frac{k^2u_0^2}{2h^2}\right) = \frac{1}{2}\beta, \tag{3}$$

it becomes

$$\left(\frac{d\dfrac{u}{u_0}}{d\theta}\right)^2 = \frac{1}{2}\beta - \left(\frac{u}{u_0}\right)^2 + \frac{1}{2}b\left(\frac{u}{u_0}\right)^4. \tag{4}$$

If the transformation

$$u = \frac{1}{r}, \qquad u_0 = \frac{1}{r_0}$$

is made, it is found that

$$\left(\frac{d\dfrac{r}{r_0}}{d\theta}\right)^2 = \frac{1}{2}b - \left(\frac{r}{r_0}\right)^2 + \frac{1}{2}\beta\left(\frac{r}{r_0}\right)^4. \tag{5}$$

Equations (4) and (5) differ only in that the constants b and β are interchanged. Consequently, any solution of Eq. (5) for r/r_0 is also a solution for its reciprocal when b and β are interchanged. Hence, if

$$r = r_0\varphi(\theta;\ b,\ \beta)$$

is a solution of Eq. (5), so also is

$$u = u_0\varphi(\theta;\ \beta,\ b)$$

a solution. That is,

$$\varphi(\theta;\ b,\ \beta)\cdot\varphi(\theta;\ \beta,\ b) = 1,$$

which is a very curious relationship.

It is evident from the form of Eq. (5) that the general solution for r/r_0 is an elliptic function of θ. The elliptic functions of Weierstrass lead to a more beautiful analysis of this problem than do the elliptic functions of Jacobi. Equation (5) is reduced to the normal form of Weierstrass by the substitutions

$$\left(\frac{r}{r_0}\right)^2 = \frac{\frac{1}{2}b}{s + \frac{1}{3}}, \qquad b\beta = \gamma,$$

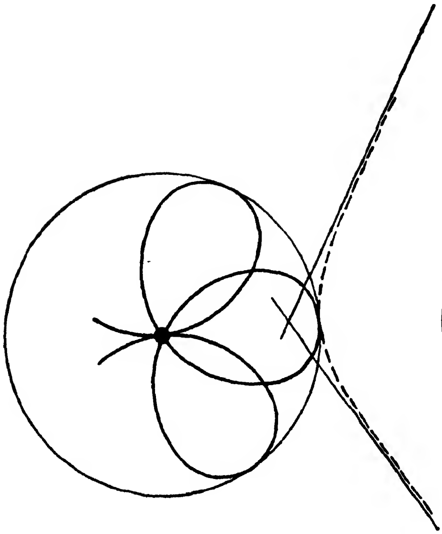
where s is the new variable. That is, Eq. (5) becomes

$$\left(\frac{ds}{d\theta}\right)^2 = 4\left[s + \frac{1}{3}\right]\left[s - \frac{1}{6} - \frac{1}{2}\sqrt{1-\gamma}\right]\left[s - \frac{1}{6} + \frac{1}{2}\sqrt{1-\gamma}\right],$$

the solution of which is

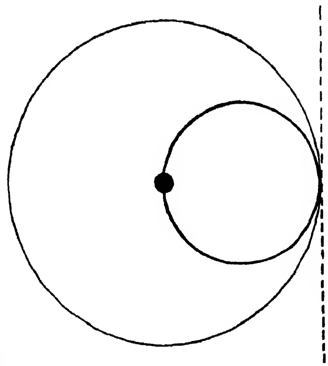
$$s = \wp(\theta - \theta_0),$$

where $\wp(\theta - \theta_0)$ is the Weierstrass *pi* function.



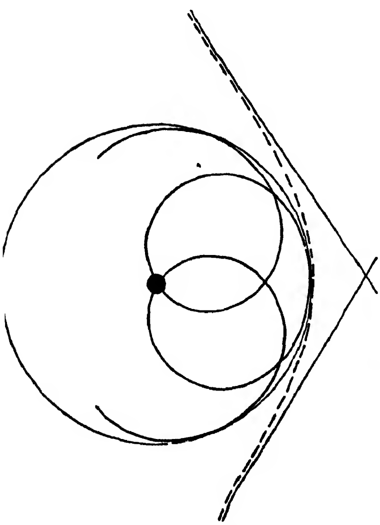
$\gamma = -\frac{5}{4}$

Fig. 156.



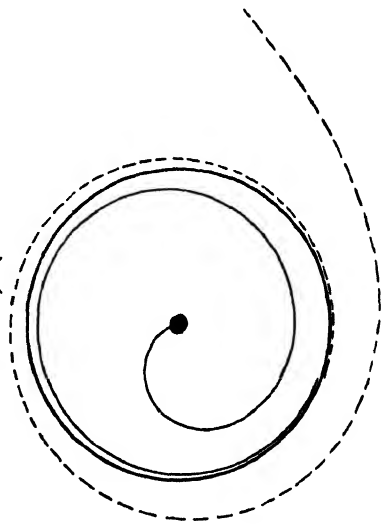
$\gamma = 0$

Fig. 157.



$\gamma = +\frac{15}{16}$

Fig. 158.



$\gamma = 1$

Fig. 159.

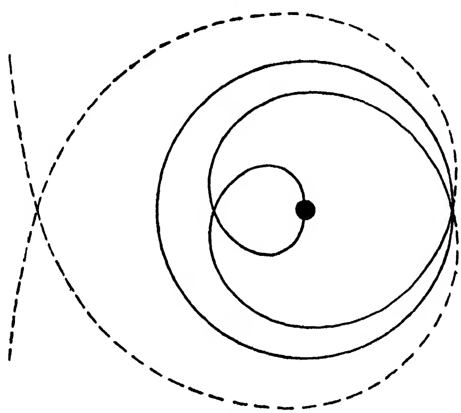
The constant γ plays a rôle somewhat similar to the eccentricity in the family of conics. It determines the shape of the orbits. Conics are divided into two classes, namely, those for which $e < 1$, and those for which $e > 1$, with the straight line, circle, and parabola as limiting cases. The present family of orbits is divided into three classes, according as

$$-\infty < \gamma < 0, \qquad 0 < \gamma < 1, \qquad 1 < \gamma < +\infty;$$

with the limiting cases

$$\gamma = -\infty, \qquad \gamma = 0, \qquad \gamma = +1, \qquad \gamma = +\infty.$$

Diagrams illustrating the different cases are given in Figs. 156 to 160, the dotted curves being the reciprocal orbits. It will



$\gamma = 1.005$

Fig. 160.

be observed that with the exception of the circle $r/r_0 = 1$, which is drawn in each diagram, every orbit passes through the origin or through infinity. Hence, with this law of attraction, a family of planets and satellites, such as our own, could not exist.

A complete analysis of this problem requires a rather full knowledge of elliptic functions, and is, therefore, out of place in the present volume. A full discussion, however, will be found in the *American Journal of Mathematics*, Vol. XXX. For the limiting cases $\gamma = 0$ and $\gamma = 1$, the problem is solvable by means of elementary functions.

311. The Limiting Case $\gamma = 0$.—If the constant γ is zero, either $b = 0$ or $\beta = 0$; and, since the solution for either is the

reciprocal of the solution for the other, it is immaterial which is chosen. Let $\beta = 0$. Then Eq. (310.5) becomes

$$\frac{d\dfrac{r}{r_0}}{d\theta} = \sqrt{\frac{1}{2}b - \left(\frac{r}{r_0}\right)^2},$$

for which the solution is

$$r = r_0\sqrt{\frac{1}{2}b} \cos (\theta - \theta_0).$$

Since $r = r_0$ for $\theta = 0$, it is evident that

$$1 = \sqrt{\frac{1}{2}b} \cos \theta_0.$$

Hence, $b \geqq 2$. If $b = 2$, then $\theta_0 = 0$, and the simplest form of the solution is

$$r = r_0 \cos \theta.$$

This is a circle which has the line joining the initial point to the origin as a diameter.

The reciprocal solution

$$r = r_0 \sec \theta$$

is a straight line. Indeed, if b is zero, it is seen from Eq. (310.3) that k^2 is zero, and therefore the force which is acting is always zero.

312. The Limiting Case $\gamma = 1$.—If the product $b\beta$ is equal to unity, the right member of Eq. (310.5) is a perfect square, and the equation can be written

$$\frac{d\dfrac{r}{r_0}}{d\theta} = \frac{1}{\sqrt{2b}}\left[\left(\frac{r}{r_0}\right)^2 - b\right],$$

for which the solution is

$$r = r_0\sqrt{b} \coth (\theta - \theta_0).$$

The orbit is, therefore, a spiral which is asymptotic to the circle $r = r_0\sqrt{b}$ on the outside. The reciprocal solution is

$$r = r_0\sqrt{\beta} \tanh (\theta - \theta_0),$$

which is a spiral which passes through the origin and is asymptotic to the circle $r = r_0\sqrt{\beta}$ on the inside.

If

$$b = \beta = 1,$$

the solution degenerates into the circle $r = r_0$, which is the only orbit which does not pass through the origin or through infinity.

Problems XXI

1. Find the central force under which a particle describes the spiral $r\theta = c$; the spiral $r = e^\theta$. *Ans.*

$$f = -\frac{mh^2}{r^3}; \qquad f = -\frac{2mh^2}{r^3}.$$

2. Show that if the particle describes the lemniscate,

$$r^2 = a^2 \cos 2\theta,$$

under the action of a central force, then

$$f = -\frac{3ma^4h^2}{r^7}.$$

3. The central force under which a particle describes the cardioid

$$r = a(1 + \cos\theta)$$

is

$$f = -\frac{3mah^2}{r^4}.$$

4. The central force under which a particle describes the circle

$$r = 2a\cos\theta$$

is

$$f = -\frac{8ma^2h^2}{r^5}.$$

5. The central force under which a particle describes the curve

$$x^4 - y^4 = a^4 \quad \text{is} \quad f = \frac{3mh^2}{4a^3}\left[-\left(\frac{a}{r}\right)^3 + \left(\frac{r}{a}\right)^5\right];$$

and for

$$x^4 + y^4 = a^4 \quad \text{it is} \quad f = \frac{3mh^2}{2a^3}\left[\left(\frac{r}{a}\right) - \left(\frac{r}{a}\right)^5\right].$$

6. If the law of the central force is,

$$f = -m\left(\frac{\mu}{r^2} + \frac{\nu}{r^3}\right), \qquad \nu < h^2,$$

the equation of the orbit has the form

$$r = \frac{a}{1 + e\cos(k\theta)},$$

which can be regarded as a conic in which the axes are rotating.

7. If the law of the force is

$$f = -m\frac{c_1 + c_2 \cos 2\theta}{r^2},$$

the orbit is an algebraic curve of the fourth degree, except when c_2 is zero, in which case it is of the second degree.

8. If the central force is

$$f = -k^2mr,$$

and if ρ is the semidiameter conjugate to r, show that the speed in the orbit is

$$s' = k\rho$$

9. A particle is attracted toward two fixed centers for each of which the force is proportional to the distance. Show that the resultant force is a central force which is directly proportional to the distance.

10. If the maximum speed in a certain elliptic harmonic motion is 3 1/4 ft. per second, and the minimum speed is 3 ft. per second, what is the eccentricity of the ellipse? *Ans.* $e = 5/13$.

11. Prove that the average value (with respect to the time) of the kinetic energy in elliptic harmonic motion is equal to the average value of the potential energy.

12. A particle moves in an ellipse under the Newtonian law of gravitation. Prove that the average value (with respect to the time) of the kinetic energy is equal to 1/2 the average value of the potential energy.

13. If σ_E and R_E are the mean density and the radius of the earth, and σ_S and R_S are the mean density and the radius of any other sphere, show that the surface gravity g_S of the sphere is

$$g_S = \frac{\sigma_S R_S}{\sigma_E R_E} g.$$

14. Assuming that the mean distance of the moon is 238,000 miles, that the radius of the earth is 3958 miles, and that the period of the moon is 27.322 days, show that a satellite of the earth which describes a circular orbit 100 miles above the surface of the earth has a period of 1 hr. 27.6 min.

15. Show that the period of a satellite which is just out of contact with the attracting sphere is independent of the size of the sphere and depends only on the density, the equation being

$$P = \sqrt{\frac{3\pi}{\kappa^2\sigma}},$$

where σ is the mean density of the sphere and κ^2 is the gravitational constant.

16. Prove that the speed of a particle falling from infinity under the Newtonian law of gravitation is

$$s' = \kappa R\sqrt{\frac{8}{3}\pi\sigma}$$

at the surface of the attracting sphere.

17. If the rate at which the radius vector sweeps over areas is the same for a circular orbit and for a parabolic orbit, show that the perihelion distance of the parabola is one-half of the radius of the circle.

18. Assuming that the equinoxes occur at the ends of the latus rectum of the orbit of the earth (which is nearly true) and that the eccentricity of the orbit of the earth is 1/60, show that the interval of time from the vernal equinox to the autumnal equinox is 7.6 days longer than the interval of time from the autumnal equinox to the vernal equinox.

19. The mean distance of the earth from the sun is the *astronomical unit.* The perihelion distance of Halley's comet is approximately zero. If its period is 75 years, what is its aphelion distance? *Ans.* 35.6 A.U.

20. The nearest fixed star is at a distance of 275,000 A.U. What is the period of a comet whose aphelion is at this distance? *Ans.* 51,000,000 years.

21. Prove that the speed in a parabolic orbit at any point is $\sqrt{2}$ times the speed in the circular orbit which passes through that point.

22. If s_a' and s_p' are the speeds at aphelion and perihelion, respectively, prove that

$$s_p' : s_a' :: 1 + e : 1 - e.$$

23. If the mass of the earth were equal to the mass of the sun and its mean distance the same as at present, what would its period be? *Ans.* 258.3 days.

24. A particle moves in an ellipse in accordance with the laws of Kepler. The points p_1 and p_2 are at the ends of a diameter, and s_1' and s_2' are the speeds of the particle at p_1 and p_2, respectively. Prove that

$$s_1's_2' = \frac{k^2}{a}.$$

25. Prove that the average value of the radius vector in Keplerian elliptic motion is

$$a\left(1 + \frac{1}{2}e^2\right) \qquad \text{with respect to the time;}$$

and

$$a\sqrt{1 - e^2} \qquad \text{with respect to the polar angle.}$$

26. Prove that the average value of the reciprocal of the radius vector with respect to the time is $1/a$, and with respect to the polar angle is $1/p$.

CHAPTER XIII

CONSTRAINED MOTION

313. Freedom and Constraint.—A particle which is free to move without any restriction is said to have three *degrees of freedom*, for its velocity at any instant can be specified by three independent coordinates; or, the position of the particle in space can be specified by means of three independent coordinates.

If a particle is free to move anywhere upon a given surface,

$$\varphi(x, y, z; t) = 0, \tag{1}$$

but cannot leave the surface, it is said to have two degrees of freedom and one *degree of constraint*. The surface, or even the equation of the surface (Eq. (1)), is called the *constraint*.

If a particle is free to move anywhere upon a given line, which is defined by the equations,

$$\varphi_1(x, y, z; t) = 0, \qquad \varphi_2(x, y, z; t) = 0, \tag{2}$$

but cannot leave the line, it is said to have one degree of freedom and two degrees of constraint, the constraints being the two equations which define the line.

It will be observed that the constraints may depend upon the time. If they do, the surface or line in which the particle is compelled to move is itself in motion, but in a perfectly definite, specified manner. If the constraints do not depend upon the time, the surface or line to which the particle is constrained is fixed in space.

The constraint may be *complete*, as when a particle is compelled to move upon the surface of a sphere by means of a light stiff rod which is fixed at one point and to which the particle is rigidly attached; or it may be *one sided*, as when the rigid rod just mentioned is replaced by a light string. In the latter case, the particle is free to move anywhere within the sphere whose radius is the length of the string between the fixed point and the particle.

In the present chapter, only fixed constraints will be considered, and, for the most part, the constraints will be complete.

I. LINEAR CONSTRAINTS

314. The Particle Has One Degree of Freedom.—The motion of a particle along a curve depends upon the forces which are acting upon it, and not at all upon the mechanical contrivance by means of which the particle is constrained to follow the curve; but, for the sake of definiteness and simplicity, the particle will be regarded as a bead which slides along a wire which has the form of the given curve.

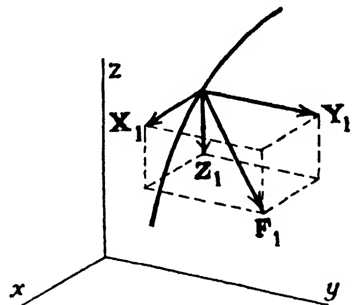


FIG. 161.

The forces which are acting upon the bead can be divided into two classes: first, the forces which act upon the bead irrespective of the wire, such as gravity, tensions of strings, magnetism, etc.; second, the action of the wire upon the bead. Let X, Y, and Z be the magnitudes of components of the resultant $\mathbf{F}$ of the forces of the first class, and X_1, Y_1, and Z_1 the components of the action of the wire $\mathbf{F}_1$ upon the bead. Then the equations of motion of the bead, of which the mass is m, are (Sec. 268)

$$\left.\begin{aligned} mx'' &= X + X_1, \\ my'' &= Y + Y_1, \\ mz'' &= Z + Z_1. \end{aligned}\right\} \tag{1}$$

The motion is just the same as though the bead were free, and the forces of constraint were added to the forces which are otherwise acting.

Notwithstanding that the equations in Eq. (1) are true, they are not independent, for the coordinates of the bead must satisfy the two equations of constraint

$$\varphi_1(x, y, z) = 0, \qquad \varphi_2(x, y, z) = 0, \tag{2}$$

which define the given curve. Indeed, Eqs. (2) can be imagined solved, say for y and z in terms of x. Then y and z can be eliminated from Eq. (1), leaving a single differential equation to be solved for x. When x is obtained as a function of t, the coordinates y and z are obtained as functions of t merely by substituting the value of x. While this process is conceivable, it is not, in general, a wise method of procedure; since the forces of constraint, in general, cannot be written down explicitly.

315. The Equation of the Curve Is Given Parametrically.—Since the particle has but one degree of freedom its coordinates can be expressed by means of a single parameter. As the parameter varies, the particle moves along the curve, so that it is necessary merely to know the value of the parameter as a function of the time in order to know the position of the particle at any desired instant.

Let the parametric equations of the line be

$$x = \omega_1(q), \qquad y = \omega_2(q), \qquad z = \omega_3(q), \tag{1}$$

where q is the parameter.

If these expressions for the coordinates are substituted in the expression for the kinetic energy (Eq. (269.2)), the following equation is obtained:

$$\frac{1}{2}m\left[\left(\frac{d\omega_1}{dq}\right)^2 + \left(\frac{d\omega_2}{dq}\right)^2 + \left(\frac{d\omega_3}{dq}\right)^2\right]q'^2 - \frac{1}{2}mv_0^2 =$$

$$\int_{x_0,y_0,z_0}^{x,y,z}[Xdx + Ydy + Zdz] + \int_{x_0,y_0,z_0}^{x,y,z}[X_1dx + Y_1dy + Z_1dz]. \tag{2}$$

The coefficient of q'^2 in the left member of this equation is a function of q alone. The first integral of the right member is the work done by the applied forces in moving the particle along the curve from the initial point x_0, y_0, z_0 to the point x, y, z; while the second integral in the right member is the work done by the forces of constraint. The components of the forces of constraint which are normal to the curve do no work, since they are perpendicular to the displacements of the bead. The tangential component, which arises from friction of the bead with the wire or to the resistance of the surrounding medium, will do work, however, unless the wire is smooth and a surrounding medium does not exist.

In what follows it will be supposed that friction and resistance do not occur. It will be supposed further that the components of the applied forces, X, Y, and Z, depend upon the position of the particle alone. They are therefore functions of q, but not functions of q' and t. Under these circumstances,

$$Xdx + Ydy + Zdz = Qdq,$$

where Q is a function of q alone; and Eq. (2) becomes

$$\frac{1}{2}m\left[\left(\frac{d\omega_1}{dq}\right)^2 + \left(\frac{d\omega_2}{dq}\right)^2 + \left(\frac{d\omega_3}{dq}\right)^2\right]q'^2 - \frac{1}{2}mv_0^2 = \int_{q_0}^{q}Qdq, \tag{3}$$

the right member of which is a function of q. If it is assumed that this function is known, Eq. (3) can be solved for q', namely,

$$q'^2 = f(q)$$

and, therefore,

$$t - t_0 = \pm \int_{q_0}^{q} \frac{dq}{\sqrt{f(q)}}. \tag{4}$$

Thus the solution of the problem is obtained by two quadratures.

316. The Applied Force is Gravity Only.—Under the hypotheses that gravity is the only force acting upon the bead aside from the constraints, that the wire is smooth, and that the x- and y-axes are horizontal and that the positive end of the z-axis is directed vertically upward, the components of the applied forces are

$$X = Y = 0, \qquad Z = -mg.$$

If s' is the speed of the bead along the curve, Eq. (315.3) becomes, after removing the factor m,

$$s'^2 - v_0^2 = -2g \int_{x_0, y_0, z_0}^{x, y, z} dz = 2g(z_0 - z). \tag{1}$$

Thus the speed of the bead depends only upon the vertical distance through which bead has fallen, and not at all upon the shape of the curve. For a given curve z is a known function of the arc s and, therefore, Eq. (1) gives the result

$$t - t_0 = \int_{s_0}^{s} \frac{ds}{\sqrt{v_0^2 + 2g(z_0 - z(s))}}. \tag{2}$$

317. The Fixed Curve is a Helix.—Suppose, for example, the given curve is the helix

$$x = a \cos \omega, \qquad y = a \sin \omega, \qquad z = b\omega, \tag{1}$$

which lies upon a vertical cylinder of radius a. Then

$$x'^2 + y'^2 + z'^2 = (a^2 + b^2)\omega'^2.$$

Let the initial values of ω and s' be

$$\omega_0 = 2n\pi, \qquad s_0' = v_0 = 0.$$

Then Eq. (316.1) becomes

$$\omega'^2 = \frac{2bg}{a^2 + b^2}(2n\pi - \omega),$$

or

$$-\frac{\omega'}{\sqrt{2n\pi - \omega}} = \sqrt{\frac{2bg}{a^2 + b^2}}.$$

The negative sign is necessary since $2n\pi - \omega$ cannot be negative, and, therefore, ω must be a decreasing function of the time.

The integral of this equation which satisfies the initial condition is

$$\sqrt{2n\pi - \omega} = \frac{1}{2}\sqrt{\frac{2bg}{a^2 + b^2}}\,t,$$

or

$$\omega = 2n\pi - \frac{bgt^2}{2(a^2 + b^2)}.$$

Hence,

$$\left.\begin{aligned} x &= a \cos \frac{bgt^2}{2(a^2 + b^2)}, \qquad y = -a \sin \frac{bgt^2}{2(a^2 + b^2)}, \\ z &= 2\pi nb - \frac{b^2gt^2}{2(a^2 + b^2)}. \end{aligned}\right\} \tag{2}$$

If the angle which the tangent to the helix makes with the xy-plane be denoted by α, then

$$\sin \alpha = \frac{b}{\sqrt{a^2 + b^2}}$$

and

$$z = 2n\pi b - \frac{1}{2} \sin^2 \alpha \cdot gt^2.$$

Thus the time required for the bead to descend through a distance z is the same as though, starting from rest, it were sliding down an inclined plane for which the angle of inclination with the horizontal was α (Sec. 243).

If for simplicity of notation, the substitution

$$\mu = \frac{bg}{a^2 + b^2},$$

is made, the equation of the hodograph is

$$x' = -a\mu t \sin \frac{1}{2}\mu t^2, \qquad y' = -a\mu t \cos \frac{1}{2}\mu t^2, \qquad z' = -b\mu t. \tag{3}$$

This is the equation of a helix which lies on a cone the generating angle of which is

$$\beta = \tan^{-1} \frac{a}{b}.$$

The radius vector ρ is proportional to the time, since

$$\rho = \sqrt{x'^2 + y'^2 + z'^2} = \sqrt{a^2 + b^2}\mu t.$$

The coils of the helix are not equidistant, however, but draw closer together as the time increases.

On differentiating Eq. (3), it is found that

$$\left.\begin{aligned} x'' &= -a\mu^2t^2 \cos \frac{1}{2}\mu t^2 - a\mu \sin \frac{1}{2}\mu t^2, \\ y'' &= +a\mu^2t^2 \sin \frac{1}{2}\mu t^2 - a\mu \cos \frac{1}{2}\mu t^2, \\ z'' &= -b\mu. \end{aligned}\right\} \qquad (4)$$

The components of the reaction of the helix on the bead are obtained by the substitution of Eq. (4) in Eq. (314.1). Since

$$X = 0, \qquad Y = 0, \qquad Z = -mg,$$

it is found that

$$X_1 = mx'', \qquad Y_1 = my'', \qquad Z_1 = +\frac{a^2g}{a^2 + b^2}.$$

Let **R** and **T** be the components of the reaction parallel to the xy-plane, but along the radius vector and perpendicular to it. Then

$$R = -a\mu^2t^2, \qquad T = -a\mu, \qquad Z_1 = +\frac{a^2g}{a^2 + b^2}. \qquad (5)$$

Thus two of the components are constant, while the third is proportional to the square of the time.

II. PENDULUMS

318. The Simple Pendulum.—The simple pendulum consists of a particle which is constrained to move without friction on the circumference of a vertical circle and which is acted upon only by gravity. Mechanically, it is approximated by a heavy bob suspended by a light rod one point of which is fixed. Let O (Fig. 162) be the point of suspension of the pendulum, and let l be its length, so that l is the radius of the circle which is described by the center of gravity of the bob.

The component of the acceleration of gravity perpendicular to the rod is $-g \sin \theta$. In polar coordinates the acceleration perpendicular to the radius vector is (Sec. 257) $r\theta'' + 2r'\theta'$. But, since the curve is a circle,

$$r = l \qquad \text{and} \qquad r' = 0;$$

hence,

$$l\theta'' = -g \sin \theta. \tag{1}$$

and this is the equation of motion of the pendulum. On multiplying through by $2\theta'$ and then integrating, there results

$$\theta'^2 = \frac{2g}{l}(\cos \theta - \cos \theta_0), \tag{2}$$

the constant of integration having been determined so that the speed vanishes when $\theta = \theta_0$. This would be the case if the pendulum were released from rest at $\theta = \theta_0$.

Since

$$\cos \theta = 1 - 2 \sin^2 \frac{1}{2}\theta,$$

Eq. (2) can be written

$$\theta' = \pm 2\sqrt{\frac{g}{l}}\sqrt{\sin^2 \frac{1}{2}\theta_0 - \sin^2 \frac{1}{2}\theta}. \tag{3}$$

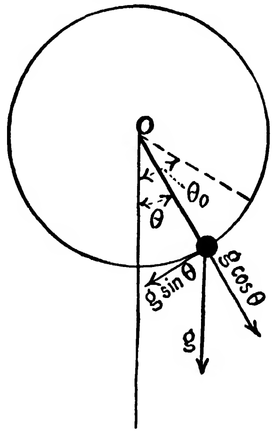


FIG. 162.

319. Case I: The Angle θ_0 is Real.—If the pendulum actually stops at some point of the circle, the angle θ_0 is real and less than π. If the pendulum does not stop at any point of the circle the angle θ_0 is equal to π or is complex. It will be supposed in the present section that

$$0 < \theta_0 < \pi,$$

and for this range of values it is convenient to make the substitution

$$\sin \frac{1}{2}\theta = \sin \frac{1}{2}\theta_0 \cdot \sin \varphi, \tag{1}$$

thus introducing a new variable φ. This substitution transforms Eq. (318.3) into

$$\varphi' = \sqrt{\frac{g}{l}}\sqrt{1 - \sin^2 \frac{1}{2}\theta_0 \sin^2 \varphi}; \tag{2}$$

and therefore, if

$$\sin^2 \frac{1}{2}\theta_0 = k^2,$$

the value of the time is given by the integral

$$t - t_0 = \sqrt{\frac{l}{g}}\int_0^{\varphi} \frac{d\varphi}{\sqrt{1 - k^2 \sin^2 \varphi}}; \tag{3}$$

t_0 being the value of t when the pendulum is at its lowest point. When the pendulum is at its highest point, φ is equal to $\pi/2$ and, therefore, the expression for the quarter period is

$$\frac{1}{4}P = \sqrt{\frac{l}{g}}\int_0^{\frac{\pi}{2}} \frac{d\varphi}{\sqrt{1 - k^2 \sin^2 \varphi}}. \tag{4}$$

This is Legendre's elliptic integral of the first kind and is usually denoted by the symbol $K(k)$. It was tabulated by Legendre for values of $k < 1$. A five-place table will be found in Peirce's "Short Table of Integrals," page 121. If k^2 is small, the integral (Eq. (4)) can be expanded in powers of k^2 by the binomial theorem, and then integrated term by term. Since

$$(1 - k^2 \sin^2 \varphi)^{-\frac{1}{2}} = 1 + \frac{1}{2}k^2 \sin^2 \varphi + \frac{1 \cdot 3}{2 \cdot 4}k^4 \sin^4 \varphi + \cdots$$
$$+ \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)} k^{2n} \sin^{2n} \varphi + \cdots,$$

and

$$\int_0^{\frac{\pi}{2}} \sin^{2n} \varphi \, d\varphi = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)} \cdot \frac{\pi}{2},$$

it follows that the value of the period P is

$$P = 2\pi\sqrt{\frac{l}{g}}\left[1 + \left(\frac{1}{2}\right)^2 k^2 + \left(\frac{1 \cdot 3}{2 \cdot 4}\right)^2 k^4 + \cdots \right.$$
$$\left. + \left(\frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)}\right)^2 k^{2n} + \cdots\right]. \tag{5}$$

The parameter k^2 increases as the amplitude of the oscillation θ_0 increases; and it is evident that P increases as k^2 increases. In this respect it differs from the period of simple harmonic motion which is independent of the amplitude of the oscillation (Sec. 251).

For infinitesimal oscillations, Eq. (318.1) reduces to

$$\theta'' = -\frac{g}{l}\theta,$$

which is the equation of simple harmonic motion; and the solution is

$$\theta = \theta_0 \sin \sqrt{\frac{g}{l}}\, t.$$

The period is

$$P_0 = 2\pi\sqrt{\frac{l}{g}},$$

and this is the value to which Eq. (5) reduces for k^2 equal to zero. The true expression for the period (Eq. (5)) shows that for finite oscillations P is always greater than the value P_0 which, on account of its simplicity, is commonly used.

320. Case II: The Angle θ_0 is Equal to π.—If the angle θ_0 is equal to π, Eq. (318.2) reduces to

$$\theta' = 2\sqrt{\frac{g}{l}}\cos\frac{\theta}{2},$$

and, therefore,

$$t - t_0 = \sqrt{\frac{l}{g}}\int_0^\theta \sec\frac{\theta}{2}d\frac{\theta}{2} = \sqrt{\frac{l}{g}}\log\tan\left(\frac{\pi}{4} + \frac{\theta}{4}\right);$$

t_0 being the value of t for θ equal to zero. For all positive values of $t - t_0$ the particle is rising, since

$$\tan\left(\frac{\pi}{4} + \frac{\theta}{4}\right) = e^{\sqrt{\frac{g}{l}}(t-t_0)}. \tag{1}$$

The pendulum approaches the vertical position asymptotically, never reaching it.

321. Case III: The Pendulum Does Not Oscillate.—If the constant of integration in Eq. (318.2) is greater than $+2g/l$, there does not exist a real angle θ_0 for which the angular speed of the pendulum vanishes, and the pendulum continues to move always in the same direction.

Let a be a positive number greater than l. Then for this case, Eq. (318.2) can be written

$$\begin{aligned} l^2\theta'^2 &= 2g(a + l\cos\theta) = 2g\left[a + l - 2l\sin^2\frac{1}{2}\theta\right] \\ &= 2g(a + l)\left(1 - k^2\sin^2\frac{1}{2}\theta\right), \end{aligned} \tag{1}$$

where

$$k^2 = \frac{2l}{a + l} < 1,$$

and k^2 decreases as a increases. For the sake of notation, let

$$n = \frac{\sqrt{2g(a + l)}}{2l}.$$

Then the integral of Eq. (1) is

$$n(t - t_0) = \int_0^\theta \frac{d\frac{\theta}{2}}{\sqrt{1 - k^2\sin^2\frac{\theta}{2}}}. \tag{2}$$

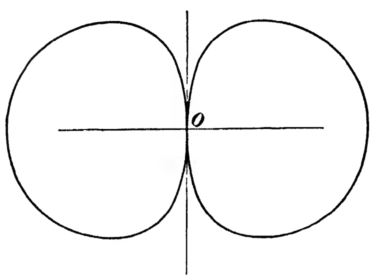
This is the same elliptic integral as before. Taking $\theta = \pi$ for the upper limit of the integral, the corresponding value of $t - t_0$ is the half period. Hence, the expression for the complete period is

$$P = \frac{\pi}{n}\left[1 + \left(\frac{1}{2}\right)^2 k^2 + \left(\frac{1 \cdot 3}{2 \cdot 4}\right)^2 k^4 + \cdots\right]. \tag{3}$$

322. The Pendulum Hodograph.—From Eq. (318.2) is derived

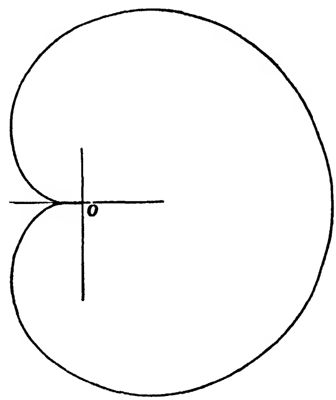
$$l\theta' = v = \sqrt{2gl}\sqrt{\cos\theta - \cos\theta_0}, \tag{1}$$

which is the equation of the hodograph in polar coordinates. The angle which the velocity vector makes with the horizontal is θ or $\theta + \pi$, the direction of the velocity changing from one to the other where the angular speed vanishes; so that in the hodograph (Eq. (1)) the angle θ is measured from a horizontal line.



Hodograph of pendulum
$\theta_0 = \pi/2$

Fig. 163.



Hodograph of pendulum.
$\theta_0 = \pi$

Fig. 164.

323. The Tension in the Pendulum Rod.—By Eq. (257.5), the acceleration along the radius vector is $r'' - r\theta'^2$, and the force necessary to produce this acceleration is $m(r'' - r\theta'^2)$. This is the pull of the rod on the bob. The pull of the bob on the rod is equal and opposite, namely, $m(-r'' + r\theta'^2)$, and this pull arises from the inertia of the bob. To this must be added the component of gravity along the radius vector, namely, $mg \cos\theta$. Since

$$r = l \qquad r'' = 0,$$

the sum of these two forces, that is, the tension in the rod, is

$$T = ml\theta'^2 + mg\cos\theta = mg(3\cos\theta - 2\cos\theta_0).$$

The tension changes into a thrust if and when

$$\cos\theta = \frac{2}{3}\cos\theta_0.$$

324. The Effect of a Resisting Medium.—In order to attain the results of the previous discussion, it would be necessary not only to free the pendulum from friction but also to swing the pendulum in a vacuum in order to free it from the effects of a resisting medium. If, however, these actions are taken into account, the equation of motion (Eq. (318.1)) becomes

$$ml\theta'' = -mg\sin\theta + R, \tag{1}$$

where R is the resistance due to friction and the medium. Since the resistance vanishes with the velocity, R must carry θ' as a factor, and in general will be a function θ' only.

First Hypothesis.—If the amplitude of the oscillation of the pendulum is small, R will probably be proportional to θ', that is,

$$R = -2mkl\theta',$$

the direction of the resistance being always opposite to the velocity of the bob. Also, for small oscillations, $\sin\theta$ can be replaced by θ, and the acceleration equation is, approximately,

$$l\theta'' + 2kl\theta' + g\theta = 0. \tag{2}$$

Since this is a linear, homogeneous, differential equation with constant coefficients, the substitution

$$\theta = e^{\lambda t}$$

reduces it to

$$le^{\lambda t}\left[\lambda^2 + 2k\lambda + \frac{g}{l}\right] = 0;$$

and, therefore,

$$\lambda = -k \pm \sqrt{k^2 - \frac{g}{l}}.$$

If the resistance is feeble, the constant k is small and the radicand is negative. Setting, therefore,

$$\mu^2 = \frac{g}{l} - k^2,$$

the two values of λ are

$$\lambda_1 = -k + i\mu, \qquad \lambda_2 = -k - i\mu, \qquad i = \sqrt{-1}.$$

Hence, the general solution of Eq. (1) is

$$\begin{aligned} \theta &= Ae^{(-k+i\mu)t} + Be^{(-k-i\mu)t} \\ &= e^{-kt}[Ae^{i\mu t} + Be^{-i\mu t}], \end{aligned}$$

where A and B are the constants of integration. But since

$$e^{i\mu t} = \cos \mu t + i \sin \mu t, \qquad e^{-i\mu t} = \cos \mu t - i \sin \mu t,$$

the solution can be written

$$\begin{aligned} \theta &= e^{-kt}[(A + B) \cos \mu t + i(A - B) \sin \mu t] \\ &= e^{-kt}[C \cos \mu t + D \sin \mu t], \end{aligned}$$

where

$$C = A + B, \qquad D = i(A - B).$$

If the pendulum is released from rest when it makes an angle θ_0 with the vertical, the initial conditions are

$$\theta = \theta_0, \qquad \theta_0' = 0,$$

and the values of C and D are

$$C = \theta_0, \qquad D = \frac{k}{\mu}\theta_0.$$

Therefore, the solution which satisfies the initial conditions is

$$\left.\begin{aligned} \theta &= \frac{\theta_0}{\mu} e^{-kt} [\mu \cos \mu t + k \sin \mu t], \\ \theta' &= -\frac{k^2 + \mu^2}{\mu} \theta_0 e^{-kt} \sin \mu t. \end{aligned}\right\} \tag{3}$$

Owing to the factor e^{-kt}, which is called the *damping factor*, the amplitude of the oscillations decreases and approaches zero as a limit. The period of the oscillations, however, is

$$P = \frac{2\pi}{\mu} = 2\pi\sqrt{\frac{l}{g - k^2 l}},$$

which is constant. It will be observed that the period is longer than when there is no resistance. Not only does the resistance steadily diminish the amplitude of the oscillations, but it also increases the period of the oscillation. The amplitudes of successive oscillations decrease like the terms of a geometric progression. If $\theta_0, \theta_1, \theta_2, \cdots$ are the values of θ for $t = 0, \frac{2\pi}{\mu}, \frac{4\pi}{\mu}, \cdots,$ then

$$\theta_0 = \theta_0, \qquad \theta_1 = \theta_0 e^{-\frac{2k\pi}{\mu}}, \qquad \theta_2 = \theta_0 e^{-\frac{4k\pi}{\mu}}, \cdots,$$

$$\theta_n = \theta_0 e^{-\frac{2nk\pi}{\mu}}, \cdots.$$

Second Hypothesis.—If the resistance is assumed to be proportional to the square of the angular speed, then

$$R = \pm\frac{1}{2}mkl\theta'^2.$$

Since the resistance is always opposite in sign from θ' and θ'^2 does not change sign with θ', it is necessary to take the minus sign when θ' is positive, that is when the pendulum is swinging toward the right (in Fig. 162), and the positive sign when θ' is negative and the pendulum is swinging toward the left. When the pendulum is swinging toward the right, Eq. (1) is

$$\theta'' + \frac{1}{2}k\theta'^2 = -\frac{g}{l}\sin\theta. \tag{4}$$

Let the independent variable be changed from t to θ. Then

$$\theta'' = \frac{d\theta'}{dt} = \frac{d\theta'}{d\theta}\cdot\frac{d\theta}{dt} = \theta'\frac{d\theta'}{d\theta} = \frac{1}{2}\frac{d\theta'^2}{d\theta},$$

and Eq. (4) can be written

$$\frac{d\theta'^2}{d\theta} + k\theta'^2 = -\frac{2g}{l}\sin\theta. \tag{5}$$

This is a linear, non-homogeneous, differential equation of the first order for θ'^2. If the right member were zero, the solution would be $\theta'^2 = Ae^{-k\theta}$. If the solution of Eq. (5) is assumed to have the form

$$\theta'^2 = Ae^{-k\theta} + B\cos\theta + C\sin\theta, \tag{6}$$

where B and C are undetermined constants, it is found by substituting Eq. (6) in Eq. (5), that the equation is satisfied if

$$B = \frac{2g}{l(k^2+1)}, \qquad C = -\frac{2kg}{l(k^2+1)}.$$

Hence, the solution of Eq. (5) is

$$\theta'^2 = Ae^{-k\theta} + \frac{2g}{l(k^2+1)}(\cos\theta - k\sin\theta). \tag{7}$$

If the resistance is feeble and the initial value of θ is small, the right member of Eq. (7) can be expanded as a power series in θ. Neglecting the third and higher powers of θ, and then determining the constant A so that θ' is zero when $\theta = -\theta_0$, Eq. (7) becomes

$$\left.\begin{aligned} \theta'^2 &= \frac{g}{l(1 + k\theta_0 + \frac{1}{2}k^2\theta_0^2)}[\theta_0^2 - k\theta_0^2\theta - (1 + k\theta_0)\theta^2], \\ &= \frac{g}{l\left(1 + k\theta_0 + \frac{1}{2}k^2\theta_0^2\right)}[\theta_0 + \theta][\theta_0 - (1 + k\theta_0)\theta]. \end{aligned}\right\} \tag{8}$$

The solution of Eq. (8) which satisfies the initial conditions

$$\theta = -\theta_0, \qquad \theta' = 0,$$

is

$$\theta = -\frac{k\theta_0^2}{2(1+k\theta_0)} - \frac{\theta_0(2+k\theta_0)}{2(1+k\theta_0)}\cos\mu t, \tag{9}$$

$$\theta' = +\sqrt{\frac{g}{l}}\frac{\theta_0(2+k\theta_0)}{2\sqrt{1+k\theta_0}\left(1+k\theta_0+\frac{1}{2}k^2\theta_0^2\right)^{\frac{1}{2}}}\sin\mu t,$$

where

$$\mu = \sqrt{\frac{g}{l}\cdot\frac{1+k\theta_0}{1+k\theta_0+\frac{1}{2}k^2\theta_0^2}}.$$

The duration of the swing for which θ' is positive is

$$T_1 = \frac{\pi}{\mu} = \pi\sqrt{\frac{l}{g}\cdot\left(1+\frac{k^2\theta_0^2}{2(1+k\theta_0)}\right)}$$

which is slightly longer than for the simple pendulum, the excess depending upon the amplitude θ_0. The value of θ at the end of the swing is

$$\theta_1 = \frac{\theta_0}{1+k\theta_0},$$

as is also evident from Eq. (8).

For the return swing, it is necessary in Eqs. (8) and (9) merely to replace $-\theta_0$ by $+\theta_1$ and change k into $-k$. At the end of the second swing, the numerical value of θ is

$$\theta_2 = \frac{\theta_1}{1+k\theta_1} = \frac{\theta_0}{1+2k\theta_0}$$

and

$$T_2 = \pi\sqrt{\frac{l}{g}\left(1+\frac{k^2\theta_0^2}{2(1+k\theta_0)(1+2k\theta_0)}\right)}.$$

In general, at the end of the n^{th} swing, the numerical values are

$$\theta_n = \frac{\theta_0}{1+nk\theta_0}$$

and

$$T_n = \pi\sqrt{\frac{l}{g}\left(1+\frac{k^2\theta_0^2}{2[1+nk\theta_0][1+(n-1)k\theta_0]}\right)}.$$

Not only does the amplitude of the swing always decrease, but the duration of the swing also decreases and approaches the half period of the simple pendulum.

325. The Cycloidal Pendulum.—For small oscillations the simple pendulum gives very nearly simple harmonic motion. It is natural to enquire, therefore, whether there exists a curve, along which the particle is constrained to move without friction, for which the motion is exactly simple harmonic under the action of gravity.

In Fig. 165, let s be the length of arc, measured from some convenient point. According to Sec. 258, the acceleration along the curve is s''. Let θ be the angle which the normal to the curve makes with the vertical. The component of the acceleration of gravity along the curve is $-g \sin \theta$, in accordance with the usual conventions. Hence, whatever the curve may be, the differential equation

$$s'' = -g \sin \theta \tag{1}$$

must be satisfied. If the motion is to be simple harmonic, the differential equation

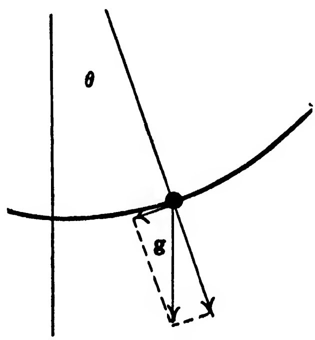


Fig. 165.

$$s'' = -k^2 s, \tag{2}$$

where k^2 is some constant, also must be satisfied. Since the left members of Eqs. (1) and (2) are the same, the right members also must be the same, and therefore

$$\sin \theta = \frac{k^2}{g} s. \tag{3}$$

Since θ is also the angle which the curve makes with the horizontal, which is here taken to be the x-axis,

$$\cos \theta = \frac{dx}{ds}, \qquad \sin \theta = \frac{dy}{ds}. \tag{4}$$

Hence, on differentiating Eq. (3) and then eliminating ds by means of Eq. (4),

$$dx = \frac{g}{k^2} \cos^2 \theta \, d\theta,$$

$$dy = \frac{g}{k^2} \sin \theta \cos \theta \, d\theta.$$

Then, by integration,

$$\left.\begin{aligned} x - x_0 &= \frac{g}{4k^2}(2\theta + \sin 2\theta), \\ y - y_0 &= -\frac{g}{4k^2} \cos 2\theta. \end{aligned}\right\} \tag{5}$$

These are the parametric equations of a cycloid with the concave side upward, and such cycloids are the only curves which satisfy the conditions. The origin will be midway between the two cusps if

$$x_0 = 0, \qquad y_0 = -\frac{g}{4k^2};$$

and with these values Eq. (5) becomes

$$\left.\begin{aligned} x &= \frac{g}{4k^2}(2\theta + \sin 2\theta), \\ y &= -\frac{g}{4k^2}(1 + \cos 2\theta). \end{aligned}\right\} \tag{6}$$

If a is the radius of the rolling circle which generates the cycloid, then (Sec. 267)

$$a = \frac{g}{4k^2}, \qquad \text{or} \qquad k^2 = \frac{g}{4a}.$$

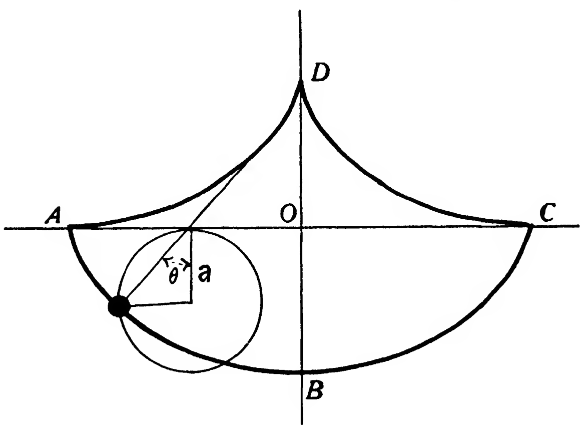


FIG. 166.

The enquiry is, therefore, answered in the affirmative, and the cycloid with the concave side upward is the curve sought. The period of the motion is (Sec. 251)

$$P = \frac{2\pi}{k} = 2\pi\sqrt{\frac{4a}{g}}, \tag{7}$$

and this is the same as the period of infinitesimal oscillations of a simple pendulum of which the length is $4a$. From the general equation for the radius of curvature of the cycloid

$$\rho = 4a \cos \theta,$$

it is seen that at its lowest point the cycloid is best fitted by a circle of radius $4a$.

Since the evolute of a cycloid ABC (Fig. 166) is a similar cycloid[1] ADC, it is evident that if the upper end of the light rod which supports the bob were a thin steel spring, fastened at D, and the arcs AD and CD were of metal and shaped to the evolute of ABC, then the bob would move on the cycloid ABC; and, theoretically, at least, the cycloidal pendulum would be realized. Pendulums of this type have been tried, but experience has decided in favor of circular pendulums of small angular oscillation, the effort being made to keep the amplitudes constant.

Since the period of the oscillation of the cycloidal pendulum is independent of the amplitude, the cycloid is a tautochrone for the force of gravity (Sec. 327).

326. The Intrinsic Equations.—Let the components of the force which is acting upon the bead along the tangent, principal normal, and binormal be f_t, f_p, and f_b; and let the magnitudes of the components of the normal reaction of the curve upon the bead be r_p and r_b. Then the intrinsic equations of the motion of the bead are (Sec. 258)

$$\left.\begin{aligned} ms'' &= f_t, \\ m\frac{s'^2}{\rho} &= f_p + r_p, \\ \text{and} \qquad 0 &= f_b + r_b. \end{aligned}\right\} \tag{1}$$

The first of these equations is closely related to the energy equation, for

$$s'' = \frac{ds'}{ds} \cdot s' = \frac{1}{2}\frac{ds'^2}{ds};$$

and, therefore,

$$\frac{1}{2}mds'^2 = f_t ds,$$

which is simply the differential of Eq. (269.2). The second and third equations are useful in determining the force which the curve exerts upon the bead.

For example, in the cycloidal motion of the previous section the integral of Eq. (325.2) is

$$s'^2 = k^2(s_0^2 - s^2),$$

s_0 being the point on the curve at which the speed vanishes.

[1] See WILLIAMSON, "Differential Calculus," p. 337.

Since

$$s = 4a \sin \theta, \qquad \rho = 4a \cos \theta,$$

it is found readily that

$$m\frac{s'^2}{\rho} = mg\,\frac{\sin^2 \theta_0 - \sin^2 \theta}{\cos \theta}.$$

From Fig. 165, it is seen that

$$f_p = -mg \cos \theta;$$

and, therefore,

$$r_p = mg\,\frac{\sin^2 \theta_0 - \sin^2 \theta + \cos^2 \theta}{\cos \theta}.$$

There is no component of force along the binormal which is perpendicular to the plane of the curve.

If the pendulum is released from rest at the points A or C, the value of θ_0 is $\pm\ \pi/2$ and

$$r_p = 2mg \cos \theta = -2f_p;$$

that is, the normal reaction of the curve is equal to twice the normal component of gravity, and in the opposite direction, if the pendulum swings over the entire cycloid.

III. TAUTOCHRONES AND BRACHISTOCHRONES

327. Tautochrones.—If a particle, starting from rest, and moving along a given curve C under the action of a given force **F** and the reaction of the curve, arrives at a fixed point O of the curve after an interval of time which is independent of the initial position of the particle on the curve, the curve is said to be a *tautochrone* for the given force; and the point O is called the *point of tautochronism.*

If F_t is the component of the force along the tangent of the curve, and X, Y, and Z are the components of the force parallel to the x-, y-, and z-axes, then the first of the intrinsic equations (Eq. (326.1)) gives

$$ms'' = F_t = \left(X\frac{dx}{ds} + Y\frac{dy}{ds} + Z\frac{dz}{ds}\right). \tag{1}$$

If s is measured from the point of tautochronism, the parametric equations of the curve can be written

$$x = \varphi_1(s), \qquad y = \varphi_2(s), \qquad z = \varphi_3(s).$$

If the force F is a function of the position only of the particle, X, Y, Z, dx/ds, dy/ds, and dz/ds will all be functions of s, and, therefore, Eq. (1) can be written

$$ms'' = \Phi(s) \tag{2}$$

where $\Phi(s)$ is some function of s. It was proved in Sec. 252, however, that the only differential equation of the form of Eq. (2) for which the period of the motion is independent of the initial position is

$$s'' = -k^2 s,$$

where k^2 is some constant, and therefore

$$X\frac{dx}{ds} + Y\frac{dy}{ds} + Z\frac{dz}{ds} = -k^2 s. \tag{3}$$

Of course,

$$\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 + \left(\frac{dz}{ds}\right)^2 = 1, \tag{4}$$

by the definitions of the quantities involved. These two differential equations must be satisfied by the curve. But since there are three quantities dx/ds, dy/ds, and dz/ds with which to satisfy them it is evident that there will be, in general, infinitely many tautochrones for a given law of force and a given point of tautochronism. It is possible to impose a third condition, and what that condition shall be, is, within limits, a matter of choice. It may be required that the tautochrone shall lie on a given surface,

$$f(x, y, z) = 0;$$

or that it shall be a tautochrone for a certain other law of force,

$$X_1 dx + Y_1 dy + Z_1 dz = -k_1^2 s,$$

or some other not impossible condition.

If there exists a potential function $U(x, y, z)$ for the given law of force, Eq. (3) is at once integrable; for

$$X\frac{dx}{ds} + Y\frac{dy}{ds} + Z\frac{dz}{ds} = \frac{dU}{ds} = -k^2 s,$$

and therefore

$$U(x, y, z) + h = -\frac{1}{2}k^2 s^2.$$

If the curve is also a tautochrone for a second law of force, for which also a potential function $U_1(x, y, z)$ exists, then likewise

$$U_1(x, y, z) + h_1 = -\frac{1}{2}k_1^2 s^2,$$

and the curve will lie upon the surface

$$k_1^2(U(x, y, z) + h) - k^2(U_1(x, y, z) + h_1) = 0.$$

328. Tautochrones on Vertical Cylinders.—It was proved in Sec. 325 that the cycloid, which is concave upward and for which the normal at the vertex is vertical, is the only tautochrone for gravity which lies in a vertical plane. If this vertical plane, including the cycloid, be wrapped around any vertical cylinder which has no corners in such a way that the original normal at the vertex coincides with one of the generators of the cylinder, the cycloid will be bent into a twisted curve on the surface of the cylinder. This twisted curve is still a tautochrone for gravity, for the bending of the vertical plane in the manner described does not alter the length of any arc of the cycloid, nor does it change the tangential component of gravity along the curve. The motion along the curve, therefore, is not altered by this bending, and the twisted curve remains a tautochrone.

Conversely, if a curve on a vertical cylinder is a tautochrone under gravity, and if the surface of the cylinder is developed on a vertical plane in such a way that its generators remain vertical, the given curve, still remaining a tautochrone, will develop into a cycloid.

329. The Brachistochrone.—Given two points A and B with A higher than B, it is required to find the curve joining A and B along which a bead will slide under the action of gravity, starting from A, with or without initial speed, in the least possible time. This famous problem was first proposed by John Bernoulli in 1696, and is, therefore, sometimes called Bernoulli's problem; at other times the required curve is called the *curve of quickest descent*, or the *brachistochrone*.

Let the x-axis be horizontal, and the z-axis vertical with the positive end directed upward. Let the xz-plane contain the two points A and B, and let the origin be at A. It is taken as evident that the required curve will lie in this plane. If s is the length of arc of the curve as measured from A, s' will be the speed of the bead along the curve. The equation of energy then gives the equation

$$s'^2 - s_0'^2 = -2gz.$$

Therefore,

$$\sqrt{2g}t = \int_A^B \frac{ds}{\sqrt{k - z}} = \int_{x_A}^{x_B} \frac{\sqrt{1 + z'^2}}{\sqrt{k - z}} dx, \tag{1}$$

where

$$k = \frac{s_0'^2}{2g}, \qquad z' = \frac{dz}{dx}.$$

Since t is to be a minimum, it is required to find a curve C for which the definite integral

$$I = \int_{x_1}^{x_2} \frac{\sqrt{1 + z'^2}}{\sqrt{k - z}} dx, \tag{2}$$

shall be a minimum, the limits of integration x_1 and x_2, which are the values of x for the points A and B, being fixed.

The problem of finding a function which minimizes a definite integral was a new type of problem at the time the brachistochrone problem was proposed, and it gave rise to that branch of mathematics which is called *the calculus of variations.*[1]

330. The Minimizing of a Certain Definite Integral.—The integral I can be written briefly[2]

$$I = \int_{x_1}^{x_2} f(z, z')dx, \qquad \text{where} \qquad f(z, z') = \frac{\sqrt{1 + z'^2}}{\sqrt{k - z}}. \tag{1}$$

The ordinates of the curve C which minimizes this integral can be written

$$z = z(x), \qquad x_1 \leqq x \leqq x_2.$$

Let $\zeta(x)$ be any continuous function of x which admits a continuous first derivative between the limits x_1 and x_2, and which vanishes for $x = x_1$ and $x = x_2$; that is,

$$\zeta(x_1) = 0, \qquad \zeta(x_2) = 0.$$

Then any curve which passes through the points A and B and is in the neighborhood of C between these points will have the form

$$z = z(x) + \epsilon\zeta(x),$$

if ϵ is a constant which is sufficiently small. The integral I, therefore, becomes

$$I(\epsilon) = \int_{x_1}^{x_2} f(z + \epsilon\zeta, z' + \epsilon\zeta')dx$$

[1] For a very readable account of the brachistochrone problem and its relation to the calculus of variations, see G. A. Bliss, "Calculus of Variations," p. 41, *The Carus Mathematical Monographs*, No. 1, 1925.

[2] In sections 330 and 331 accents indicate *differentiation with respect to x.*

with the understanding that $I(0)$ is a minimum value of $I(\epsilon)$. By the usual principles of maxima and minima, the first derivative of $I(\epsilon)$ with respect to ϵ must vanish for $\epsilon = 0$, that is,

$$\left.\frac{dI}{d\epsilon}\right|_{\epsilon=0} = \int_{x_1}^{x_2}\left(\frac{\partial f}{\partial z}\zeta + \frac{\partial f}{\partial z'}\zeta'\right)dx = 0. \tag{2}$$

By differentiating the product $\zeta\int_{x_1}^{x}\frac{\partial f}{\partial z}dx$ with respect to x, it is easily found that

$$\zeta\frac{\partial f}{\partial z} \equiv \frac{d}{dx}\left[\zeta\int_{x_1}^{x}\frac{\partial f}{\partial z}dx\right] - \zeta'\int_{x_1}^{x}\frac{\partial f}{\partial z}dx;$$

and therefore Eq. (2) can be written

$$\left.\frac{dI}{d\epsilon}\right|_{\epsilon=0} = \int_{x_1}^{x_2}\left[\zeta'\frac{\partial f}{\partial z'} - \zeta'\int_{x_1}^{x}\frac{\partial f}{\partial z}dx + \frac{d}{dx}\left(\zeta\int_{x_1}^{x}\frac{\partial f}{\partial z}dx\right)\right]dx;$$

or, since

$$\int_{x_1}^{x_2}\frac{d}{dx}\left(\zeta\int_{x_1}^{x}\frac{\partial f}{\partial z}dx\right)dx = 0,$$

by virtue of the fact that

$$\zeta(x_1) = 0, \qquad \zeta(x_2) = 0,$$

the condition that $I(0)$ shall be a minimum becomes

$$\left.\frac{dI}{d\epsilon}\right|_{\epsilon=0} = \int_{x_1}^{x_2}\left[\frac{\partial f}{\partial z'} - \int_{x_1}^{x}\frac{\partial f}{\partial z}dx\right]\zeta' dx = 0.$$

This condition must hold for every function ζ'. Therefore,

$$\frac{\partial f}{\partial z'} - \int_{x_1}^{x}\frac{\partial f}{\partial z}dx = c;$$

and, by differentiation with respect to x,

$$\frac{d}{dx}\cdot\frac{\partial f}{\partial z'} - \frac{\partial f}{\partial z} \equiv \frac{\partial^2 f}{\partial z\partial z'}z' + \frac{\partial^2 f}{\partial z'^2}z'' - \frac{\partial f}{\partial z} = 0, \tag{3}$$

assuming, of course, that the minimizing curve C has a second derivative z''. This is known as Euler's differential equation, since it was derived by Euler in 1744.

Now

$$\frac{d}{dx}\left(z'\frac{\partial f}{\partial z'} - f\right) \equiv z'\left(\frac{\partial^2 f}{\partial z\partial z'}z' + \frac{\partial^2 f}{\partial z'^2}z'' - \frac{\partial f}{\partial z}\right); \tag{4}$$

and as this vanishes, by the preceding equation, it follows that

$$z'\frac{\partial f}{\partial z'} - f = \text{constant}. \tag{5}$$

331. Application to the Brachistochrone.—The argument which has been outlined above depends essentially upon the fact that $f(z, z')$ does not contain x explicitly, and therefore it holds for a wider class of problems than the one which is being discussed at the moment. For the brachistochrone (Eq. (330.1)),

$$f = \frac{\sqrt{1 + z'^2}}{\sqrt{k - z}}$$

and

$$\frac{\partial f}{\partial z'} = \frac{z'}{\sqrt{(k - z)(1 + z'^2)}}.$$

Hence, by Eq. (330.5),

$$f - z'\frac{\partial f}{\partial z'} = \frac{1}{\sqrt{(k - z)(1 + z'^2)}} = \frac{1}{\sqrt{2a}}, \tag{1}$$

the constant being given the form $1/\sqrt{2a}$ for convenience. From Eq. (1), it is found that z satisfies the differential equation

$$z'^2 = \frac{2a - (k - z)}{(k - z)}. \tag{2}$$

In order to solve this differential equation, let a new variable θ be introduced by the relation

$$k - z = 2a \cos^2 \theta = a(1 + \cos 2\theta). \tag{3}$$

If it is not forgotten that z' is the derivative of z with respect to x, it is readily verified that this substitution transforms Eq. (2) into the equation

$$\frac{dx}{d\theta} = 2a(1 + \cos 2\theta);$$

and therefore the two coordinates x and z are given parametrically by the two equations

$$\left.\begin{aligned} x - x_0 &= a(2\theta + \sin 2\theta), \\ z - k &= -a(1 + \cos 2\theta). \end{aligned}\right\} \tag{4}$$

On comparing these equations with Eq. (325.6), it is seen that the minimizing curve C, that is, the brachistochrone, also is a cycloid with the tangent at the vertex horizontal and the curve concave upward, just as in the tautochrone. The horizontal line on which the circle rolls is at a distance k above the point A.

332. The Required Cycloid Always Exists.—Naturally, the question arises, "Does there exist a cycloid of the type described

which connects the points A and B; and if so, is it unique?" The determination of the constants a and x_0 of Eq. (331.4), for a given pair of points A and B, involves the solution of a transcendental equation, which is laborious; but it is not difficult to see geometrically that the answer is in the affirmative.

In Fig. 167, let A and B be the two given points, and let $AD = k$. If the cycloid exists, the horizontal line CDG is its base. Draw the line BE perpendicular to the base. The quadrilateral $ABED$ is a trapezoid. For simplicity of language it will be said that the problem is to circumscribe the trapezoid $ABED$ with a cycloid. Let the problem be inverted: Given a cycloid $F_1AB_1G_1$ for which the radius of the rolling circle is unity. Can a trapezoid

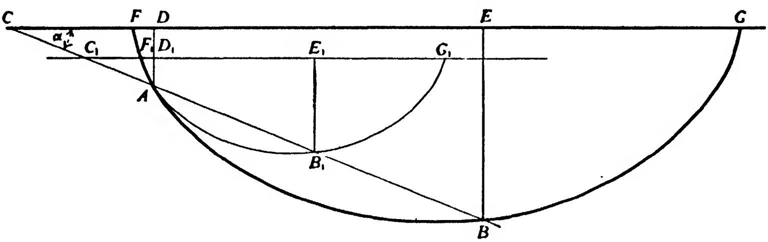


FIG. 167.

similar to $ABED$ be inscribed in it? Assume, for the moment, that it can and that the solution is unique. Let $AB_1E_1D_1$ be the required trapezoid. In the plane of $ABED$, place the lines AB_1 and AD_1 in coincidence with the lines AB and AD, respectively. With A as a center, let the plane of the cycloid $F_1AB_1G_1$ be stretched in all directions in the ratio AB/AB_1. Then the points B_1, G_1, and D_1 will coincide with the points B, G, and D, respectively and the cycloid $F_1AB_1G_1$ will be transformed into the cycloid $FABG$ which is the cycloid required, since its base is horizontal and it passes through the given points A and B.

It remains to be shown that a trapezoid similar to $ABED$ can be inscribed in the unit cycloid and that (aside from a reflection in the axis of symmetry of the cycloid) there is only one. Draw a line making an angle α with the base F_1G_1 equal to the angle BCG. This line will cut the cycloid in two points, say A and B_1. Draw the perpendiculars to the base AD_1 and B_1E_1. Then the trapezoid $AB_1E_1D_1$ will be similar to the trapezoid $ABED$ if

$$\frac{AD_1}{D_1E_1} = \frac{AD}{DE}.$$

Let C_1 be the vertex of the angle α. If the point C_1 coincides with the point F_1, then $AD_1 = 0$ and $D_1E_1 \neq 0$. Hence, the ratio AD_1/D_1E_1 is equal to zero. Let the point C_1 move toward the left. The perpendicular AD_1 increases steadily, while the distance D_1E_1 decreases steadily and has the limit zero, which occurs when the points A and B_1 coincide. Hence, the ratio AD_1/D_1E_1 increases steadily and has no upper limit. It passes the ratio AD/DE once and only once. Hence, the problem admits one and only one solution.

The problem of the brachistochrone admits of a great variety of generalizations. To find, for example, the curve of quickest descent from a fixed point to a fixed surface under the action of gravity, or from one fixed surface to another, brachistochrones which are restricted to a given surface, brachistochrones under a general law of force, etc. The solutions of these problems require a deeper knowledge of the calculus of variations than the student is assumed to possess, and the problem will not be pursued further here.

IV. SURFACE CONSTRAINTS

333. The Motion of a Particle Constrained to a Surface.—If a particle of mass m is constrained to move on a fixed, smooth surface, it will be acted upon not only by the applied forces, of which the resultant is **F** with components X, Y, and Z, but also by the reaction of the surface **R** (Fig. 168), of which the components are R_x, R_y, and R_z. As these are the only forces which are acting, the equations of motion are

$$\left.\begin{aligned} mx'' &= X + R_x, \\ my'' &= Y + R_y, \\ mz'' &= Z + R_z. \end{aligned}\right\} \qquad (1)$$

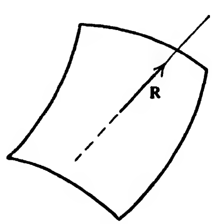


Fig. 168.

Since the surface is assumed to be smooth, its reaction is in the line of the normal to the surface through the particle. Let the equation of the surface be

$$f(x, y, z) = 0.$$

Then the equations of the tangent plane and of the normal at the point x, y, z are, respectively,

$$\frac{\partial f}{\partial x}(\xi - x) + \frac{\partial f}{\partial y}(\eta - y) + \frac{\partial f}{\partial z}(\zeta - z) = 0 \qquad \text{(tangent plane)},$$

$$\frac{\xi - x}{\dfrac{\partial f}{\partial x}} = \frac{\eta - y}{\dfrac{\partial f}{\partial y}} = \frac{\zeta - z}{\dfrac{\partial f}{\partial z}} \qquad \text{(normal)},$$

in which ξ, η, and ζ are the running coordinates and $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$, and $\frac{\partial f}{\partial z}$ are the values of the derivatives of f at the point x, y, z. The direction cosines of the normal, therefore are proportional to these derivatives; that is,

$$\frac{\partial f}{\partial x} : \frac{\partial f}{\partial y} : \frac{\partial f}{\partial z} :: \cos \widehat{nx} : \cos \widehat{ny} : \cos \widehat{nz}.$$

The magnitude of the reaction $\mathbf{R}$ is unknown; but the magnitudes of its projections upon the x-, y-, and z-axes are proportional to the direction cosines, and therefore

$$R_x : R_y : R_z :: \frac{\partial f}{\partial x} : \frac{\partial f}{\partial y} : \frac{\partial f}{\partial z}.$$

If λ is the factor of proportionality, Eq. (1) becomes

$$\left.\begin{aligned} mx'' &= X + \lambda\frac{\partial f}{\partial x}, \\ my'' &= Y + \lambda\frac{\partial f}{\partial y}, \\ mz'' &= Z + \lambda\frac{\partial f}{\partial z}. \end{aligned}\right\} \tag{2}$$

Actually,

$$\lambda = \frac{R}{\sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + \left(\frac{\partial f}{\partial z}\right)^2}},$$

but, since R is unknown, so also is λ.

The equations of Eq. (2) are true, but they are not independent, since the coordinates x, y, and z must satisfy the equation of the surface

$$f(x, y, z) = 0,$$

and the equation derived from it by differentiation with respect to t, namely,

$$\frac{\partial f}{\partial x}x' + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial z}z' = 0. \tag{3}$$

Equation (3) can be interpreted as saying that the motion at any instant lies in the tangent plane; or, if it is multiplied by λdt, it becomes

$$R_x dx + R_y dy + R_z dz = 0,$$

which states that the work done by the reaction of the surface is zero, which is obvious otherwise from the fact that the displacement of the particle is always perpendicular to the reaction.

Since motion along a curve can be regarded as motion along two mutually intersecting surfaces, the above discussion shows that the equations of motion for a particle which is constrained to move along a given curve can be written

$$\left.\begin{aligned} mx'' &= X + \lambda_1\frac{\partial f_1}{\partial x} + \lambda_2\frac{\partial f_2}{\partial x}, \\ my'' &= Y + \lambda_1\frac{\partial f_1}{\partial y} + \lambda_2\frac{\partial f_2}{\partial y}, \\ mz'' &= Z + \lambda_1\frac{\partial f_1}{\partial z} + \lambda_2\frac{\partial f_2}{\partial z}; \end{aligned}\right\} \tag{4}$$

the given curve being the intersection of the two surfaces

$$f_1(x, y, z) = 0, \qquad f_2(x, y, z) = 0,$$

and λ_1 and λ_2 being proportional, respectively, to the reactions of the two surfaces.

334. The Energy Integral.—If the first equation of Eq. (333.1) is multiplied by x', the second by y', the third by z', and the three equations are then added, there results

$$m(x'x'' + y'y'' + z'z'') = Xx' + Yy' + Zz',$$

since the coefficient of λ vanishes, by Eq. (333.3). Therefore, by integration,

$$\frac{1}{2}m(x'^2 + y'^2 + z'^2) = \frac{1}{2}ms'^2 = \int (Xdx + Ydy + Zdz) + C. \tag{1}$$

This is the energy equation, and it will be observed that it does not contain the unknown reaction R.

If the applied forces are derived from a potential function $U(x,y,z)$, then

$$Xdx + Ydy + Zdz = dU,$$

and Eq. (1) becomes

$$\frac{1}{2}ms'^2 = U + C, \tag{2}$$

just as though no constraints existed.

335. The Intrinsic Equations.—Notwithstanding that the trajectory of the particle lies upon a given surface, the intrinsic equations of Sec. 326 are still true, namely,

$$ms'' = f_t, \qquad m\frac{s'^2}{\rho} = f_p + r_p, \qquad 0 = f_b + r_b, \tag{1}$$

where the letters have the same significance as before. Let P be the position of the particle on the surface (Fig. 169) and let the plane tangent to the surface at P be drawn. Let the line AB be the tangent to the trajectory, with s increasing in the direction from A toward B. Let the plane which is normal to the surface at P and in which the line AB lies be drawn, and let C_n be the intersection of the surface with this normal plane. Finally, let the osculating plane of the trajectory at P be drawn, and let the angle between the osculating plane and the normal plane be θ.

The orthogonal projection of the trajectory upon the tangent plane is denoted by C_0. The principal radius of curvature of the

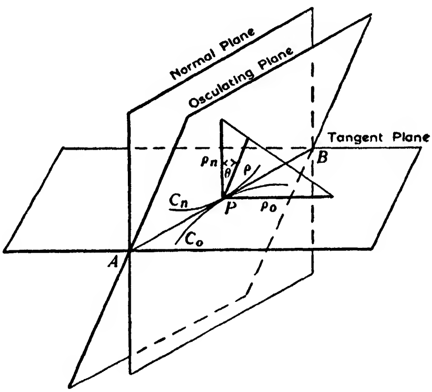


FIG. 169.

trajectory is denoted by ρ, and the radii of curvature of C_n and C_0 by ρ_n and ρ_0, respectively. The components of the applied force in the directions of the principal normal and the binormal (positive toward the right of the osculating plane in the diagram) of the trajectory are f_p and f_b. Let the components in the directions ρ_n and ρ_0 be f_n and f_0. Let r be the reaction of the surface on the particle, which, since it is normal to the surface, is directed along ρ_n. Let r_p and r_b be its components along the principal normal and along the binormal. Then

$$r \cos \theta = r_p \qquad -r \sin \theta = r_b.$$

Also,

$$f_n = f_p \cos \theta - f_b \sin \theta,$$
$$f_0 = f_p \sin \theta + f_b \cos \theta.$$

If the second and third equations of Eq. (1) are multiplied by $+\cos\theta$ and $-\sin\theta$, respectively, and added; then multiplied by $+\sin\theta$ and $+\cos\theta$ and added, there results

$$m\frac{s'^2}{\rho}\cos\theta = f_n + r,$$

$$m\frac{s'^2}{\rho}\sin\theta = f_0.$$

But it is known from the geometry of surfaces[1] that

$$\frac{\cos\theta}{\rho} = \frac{1}{\rho_n}, \qquad \frac{\sin\theta}{\rho} = \frac{1}{\rho_0}.$$

The intrinsic equations can, therefore, be written in the form

$$ms'' = f_t, \qquad m\frac{s'^2}{\rho_n} = f_n + r, \qquad m\frac{s'^2}{\rho_0} = f_0. \tag{2}$$

The second of these equations gives the important result that the normal reaction r of the surface at the point P can be computed if the velocity at P is given; since, for a given surface and applied force, ρ_n and f_n can be computed.

336. Applications of the Intrinsic Equations.—If a surface is deformed in such a way that curves drawn on the surface are not altered in length, then the *radius of geodesic curvature* which has been denoted by ρ_0 also remains unaltered in length. Hence, if the surface is so deformed and if the applied force also is changed in such a way that the component in the tangent plane is unaltered, the first and third of the intrinsic equations of Eq. (335.2) remain unaltered. The radius of curvature ρ_n of course, is altered and, in general, therefore, the normal reaction of the surface is altered; but this does not affect the motion of the particle.

If, for example, a particle moves on a vertical cylinder, the cross-section of which is a curve without corners, the path which is described upon the cylinder by a particle under the action of gravity will become a parabola with a vertical axis, if the cylinder is cut along a generator and rolled out upon a vertical plane.

If a particle describes a path upon a cone, of which the axis is vertical, under the action of gravity, and if the cone is cut along a generator and rolled out upon a horizontal plane, the path of

[1] Meusnier's theorem. See GOURSAT-HEDRICK, "Mathematical Analysis," vol. I, p. 497.

the particle will become a curve which could be described by the particle under the action of a central force of constant magnitude.

If a particle describes a path upon a cone under the action of a central force, the center of which is at the apex of the cone, the curve described when the surface of the cone is rolled out upon a plane will be a possible trajectory for a particle moving in the plane under the same law of force.

337. Geodesics.—The simplest case of motion on a surface that can arise is that for which the applied force is always zero. For this case, the equations of motion (333.1)) are

$$mx'' = \lambda\frac{\partial f}{\partial x}, \qquad my'' = \lambda\frac{\partial f}{\partial y}, \qquad mz'' = \lambda\frac{\partial f}{\partial z}; \tag{1}$$

and the energy integral (Eq. (334.2)) is

$$\frac{1}{2}ms'^2 = \text{constant}; \tag{2}$$

therefore the speed is constant. The curves described under these conditions are called the *geodesics* of the surface. They correspond to the straight lines in a plane. If two points on a surface are not too far apart, the shortest line on the surface which joins them is the geodesic which passes through them. They are also the curves assumed by a stretched string on the surface.

The intrinsic equations (Eq. (335.2)) become

$$ms'' = 0, \qquad m\frac{s'^2}{\rho_n} = r, \qquad m\frac{s'^2}{\rho_0} = 0;$$

the first of which states that the speed of the particle is constant; the second, that the normal reaction of the surface is inversely proportional to the radius of curvature of the surface which lies in the normal plane through the tangent to the curve; and the third, that ρ_0 is infinite, which is a characteristic property of geodesics.

Since the reaction of the surface is the only force which is acting, and since the acceleration of the particle always lies in its osculating plane, it is evident that the principal radius of curvature has the same direction as the normal to the surface. Therefore, ρ_n is the principal radius of curvature of the trajectory.

338. Geodesics on a Surface of Revolution.—In general, it is a very difficult matter to determine the geodesic curves on a surface but, in the case that the surface is one of revolution, the problem can always be reduced to a quadrature.

Let the equation of the surface be expressed parametrically by the equations

$$x = r \cos \theta, \qquad y = r \sin \theta, \qquad z = \Phi(r),$$

so that the z-axis is the axis of revolution. The parameters r and θ are simply the polar coordinates of the projection of a point of the surface upon the xy-plane. Since no force is acting, other than the constraint, the speed is constant. Let

$$\varphi = \frac{\partial \Phi}{\partial r}.$$

Then

$$s'^2 = r^2\theta'^2 + (1 + \varphi^2)r'^2 = v_0^2. \tag{1}$$

Furthermore, since the normal to the surface always passes through the z-axis, the moment of momentum of the particle with respect to the z-axis is constant. That is,

$$r^2\theta' = h. \tag{2}$$

Bearing in mind that

$$r' = \theta' \frac{dr}{d\theta},$$

it is found that Eq. (1) can be written

$$\left[r^2 + (1 + \varphi^2)\left(\frac{dr}{d\theta}\right)^2\right]\frac{h^2}{r^4} = v_0^2.$$

On substituting l^2 for h^2/v_0^2 and then solving, there results

$$l\frac{dr}{d\theta} = \pm\sqrt{\frac{r^4 - r^2l^2}{1 + \varphi^2}};$$

and therefore

$$\theta - \theta_0 = l\int\sqrt{\frac{1 + \varphi^2}{r^4 - r^2l^2}}dr. \tag{3}$$

Thus θ is determined as a function of r by a quadrature, the sign before the radical being determined by the initial condition. Since z is given as a function of r originally, the equations of the geodesic are given as a function of the parameter r.

The constant h is the moment of the velocity of the particle with respect to the z-axis. Let i be the angle which the geodesic makes with the meridian of the surface at any point. The velocity can be resolved into two components: $v \cos i$ along the meridian, and $v \sin i$ along the parallel perpendicular to the meridian. The moment of the component along the meridian $v \cos i$ with respect to the z-axis is zero. The moment of the

component along the parallel $v \sin i$ with respect to the z-axis is $rv \sin i$. Therefore,

$$rv \sin i = h;$$

but since v is a constant v_0 and h/v_0 is equal to l, this equation reduces to

$$r \sin i = l,$$

an equation which is due to Clairaut.

339. Motion on the Surface of a Sphere.—In terms of the spherical coordinates, the accelerations along the radius vector and in two directions perpendicular to it are (Eq. (263.2))

$$\begin{aligned} \alpha_r &= r'' - r\varphi'^2 - r\theta'^2 \cos^2 \varphi, \\ \alpha_\varphi &= r\varphi'' + 2r'\varphi' + r\theta'^2 \sin \varphi \cos \varphi, \\ \alpha_\theta &= r\theta'' \cos \varphi - 2r\varphi'\theta' \sin \varphi + 2r'\theta' \cos \varphi, \end{aligned}$$

where φ is the latitude and θ is the longitude on the sphere.

If a particle is constrained to move on the surface of a sphere of radius a, and if the center of the sphere is taken as the origin of coordinates, the radius vector r is a constant and therefore r' and r'' are always zero. Hence, if the applied force f is resolved into rectangular components f_r, f_φ, and f_θ, in which f_r is positive if directed away from the origin, f_φ is positive in the direction toward the north pole, and f_θ is positive in the direction of increasing longitudes, the equations of motion for a particle of mass m constrained to the surface of a smooth sphere are

$$\left.\begin{aligned} m(-a\varphi'^2 - a\theta'^2 \cos^2 \varphi) &= f_r + R, \\ m(a\varphi'' + a\theta'^2 \sin \varphi \cos \varphi) &= f_\varphi, \\ m(a\theta'' \cos \varphi - 2a\varphi'\theta' \sin \varphi) &= f_\theta, \end{aligned}\right\} \tag{1}$$

where R is the reaction of the sphere on the particle. The last two of these equations depend only upon the two variables φ and θ and their derivatives. For a given force f and given initial conditions, these two equations are sufficient to determine φ and θ as functions of t. When the motion is known, the first equation suffices to determine the reaction R.

If the second equation is multiplied by φ', the third by $\theta' \cos \varphi$ and they are then added, there results

$$ma(\varphi'\varphi'' + \theta'\theta'' \cos^2 \varphi - \varphi'\theta'^2 \sin \varphi \cos \varphi) = f_\varphi \cdot \varphi' + f_\theta \cos \varphi \cdot \theta';$$

and, on integrating,

$$\frac{1}{2}ma(\varphi'^2 + \theta'^2 \cos^2 \varphi) = \int (f_\varphi d\varphi + f_\theta \cos \varphi d\theta) + C. \tag{2}$$

The first equation then gives

$$-R = C + f_r + \int (f_\varphi d\varphi + f_\theta \cos\varphi d\theta).$$

Since, however,

$$s'^2 = a^2(\varphi'^2 + \theta'^2 \cos^2\varphi),$$

a simpler expression is

$$R = -\left(f_r + m\frac{s'^2}{a}\right).$$

340. The Spherical Pendulum.—If a heavy bob, attached to a light rod, is suspended from a fixed point in such a way that it is free to swing in any vertical plane through the point of suspension, it is called a spherical pendulum, since the bob remains always on the surface of a fixed sphere whose center is at the point of suspension.

Let the origin be taken at the point of suspension with the z-axis directed upward. The applied force which is acting upon the bob is its weight mg. On resolving this force along the radius vector (Fig. 170) and along two mutually perpendicular directions tangent to the sphere, it is found that

$$f_r = -mg\sin\varphi, \qquad f_\varphi = -mg\cos\varphi, \qquad f_\theta = 0.$$

The equations of motion (Eq. (339.1)), therefore, are

$$\left.\begin{aligned} \varphi'^2 + \theta'^2\cos^2\varphi &= \frac{g}{a}\cos\varphi - \frac{R}{ma}, \\ \varphi'' + \theta'^2\sin\varphi\cos\varphi &= -\frac{g}{a}\cos\varphi, \\ \theta''\cos\varphi - 2\varphi'\theta'\sin\varphi &= 0; \end{aligned}\right\} \quad (1)$$

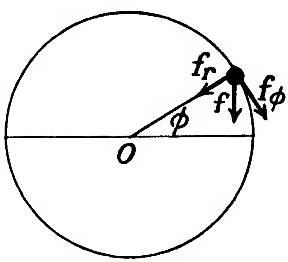


FIG. 170.

for which the energy integral (Eq. (339.2)) is

$$\varphi'^2 + \theta'^2\cos^2\varphi = -\frac{2g}{a}\sin\varphi + C. \quad (2)$$

If the last equation of Eq. (1) is multiplied by $\cos\varphi$, it becomes an exact differential, the integral of which is

$$\theta'\cos^2\varphi = h. \qquad (3)$$

This is merely the areas integral in spherical coordinates. Indeed, since the reaction of the sphere always passes through the z-axis and the applied force is always parallel to the z-axis, it is evident that the moment of all of the forces with respect to the z-axis is zero, and therefore an areas integral exists. If

the constant h is not zero, this integral shows that the meridian plane which passes through the pendulum turns always in the same direction. If h is zero, θ is a constant and the spherical pendulum reduces to the simple pendulum.

341. The Coordinate z is an Elliptic Function of θ and of t.—The result of eliminating θ' between Eqs. (340.2) and (340.3) is the equation

$$\varphi'^2 = C - \frac{2g}{a}\sin\varphi - \frac{h^2}{\cos^2\varphi}, \tag{1}$$

which defines φ as a function of the time. The time can be eliminated from this equation by means of Eq. (340.3), giving the differential equation of the path,

$$\frac{d\varphi}{d\theta} = \pm\frac{\cos\varphi}{h}\sqrt{C\cos^2\varphi - \frac{2g}{a}\sin\varphi\cos^2\varphi - h^2}; \tag{2}$$

and if this equation is multiplied through by $a\cos\varphi$, and then z is substituted for $a\sin\varphi$, it becomes

$$\frac{dz}{d\theta} = \pm\frac{a^2 - z^2}{ha^2}\sqrt{\left(C - \frac{2g}{a^2}z\right)(a^2 - z^2) - h^2a^2}. \tag{3}$$

Since the polynomial under the radical sign is of the third degree, z is an elliptic function of θ.

On setting, successively, in the radicand z equal to $-\infty$, $-a$, $+a$, and $+\infty$, the corresponding values of the radicand are

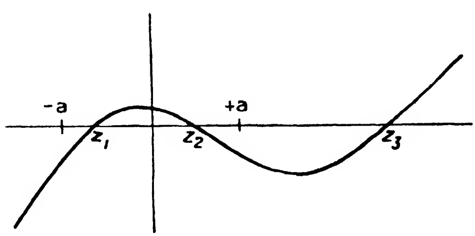


Fig. 171.

$-\infty$, $-h^2a^2$, $-h^2a^2$, and $+\infty$. Thus one root, z_3, always lies between $+a$ and $+\infty$. Since the only values of z which are admissible in the problem are those for which

$$-a \leqq z \leqq +a,$$

and since, for real values of the derivative, the radicand must be positive for such values of z as actually occur (Fig. 171), it follows

that there are two real roots between $-a$ and $+a$. Let these two roots be z_1 and z_2. By the theory of equations,

$$\left.\begin{aligned} z_1 + z_2 + z_3 &= \frac{Ca^2}{2g}, \\ z_1z_2 + z_2z_3 + z_3z_1 &= -a^2, \\ z_1z_2z_3 &= \frac{h^2 - C}{2g}a^4; \end{aligned}\right\} \tag{4}$$

and, therefore, from the second equation,

$$z_3 = -\frac{a^2 + z_1z_2}{z_1 + z_2}.$$

The corresponding values of the constants C and h^2 are

$$C = \frac{2g}{a^2}\frac{z_1^2 + z_1z_2 + z_2^2 - a^2}{z_1 + z_2},$$

$$h^2a^2 = -\frac{2g}{a^2}\frac{(a^2 - z_1^2)(a^2 - z_2^2)}{z_1 + z_2}.$$

Since h^2a^2, $a^2 - z_1^2$, and $a^2 - z_2^2$ are certainly positive if h is not zero, it follows from the expression for h^2a^2 that $z_1 + z_2$ is always negative. Interpreted geometrically, this means that there are two circles of latitude on the sphere between which the bob always lies, and that the circle of latitude which is halfway between these two always lies below the equator. On introducing the roots, z_1, z_2, and z_3, Eq. (3) takes the form

$$\begin{aligned} \frac{dz}{d\theta} = \pm\sqrt{\frac{-(z_1 + z_2)}{(a^2 - z_1^2)(a^2 - z_2^2)}} \times \\ \frac{a^2 - z^2}{a}\sqrt{(z - z_1)(z - z_2)\left(z + \frac{a^2 + z_1z_2}{z_1 + z_2}\right)}; \end{aligned} \tag{5}$$

and, similarly,

$$\frac{dz}{dt} = \pm\frac{\sqrt{2g}}{a}\sqrt{(z - z_1)(z - z_2)\left(z + \frac{a^2 + z_1z_2}{z_1 + z_2}\right)}; \tag{6}$$

so that z is an elliptic function of t also.

342. The Integration for z as a Function of t.—If the roots are taken in the order

$$z_1 < z_2 < z_3,$$

the substitution

$$z - z_1 = (z_2 - z_1)v^2,$$

where v is a new variable, reduces Eq. (341.6) to the first normal form of Legendre, namely,

$$\lambda t = \int_0^v \frac{dv}{\sqrt{(1 - v^2)(1 - k^2v^2)}},$$

where

$$k^2 = \frac{z_2 - z_1}{z_3 - z_1} \qquad \lambda = \frac{\sqrt{2g(z_3 - z_1)}}{2a}.$$

Therefore,

$$v = \operatorname{sn} \lambda t$$

and

$$z = z_1 + (z_2 - z_1)\operatorname{sn}^2 \lambda t.$$

The complete period of an oscillation for z is

$$\frac{2}{\lambda}\int_0^1 \frac{dv}{\sqrt{(1 - v^2)(1 - k^2v^2)}} = \frac{2}{\lambda}K(k),$$

where K is the same function of k that occurred for the simple pendulum (Sec. 319). The complete period for the pendulum, however, is twice this value, or

$$\text{period} = \frac{4}{\lambda}K(k),$$

since the pendulum rises and falls twice in each complete oscillation. It is interesting to note that

$$\sqrt{z - z_1} = \sqrt{z_2 - z_1}\operatorname{sn} \lambda t,$$
$$\sqrt{z_2 - z} = \sqrt{z_2 - z_1}\operatorname{cn} \lambda t,$$

and

$$\sqrt{z_3 - z} = \sqrt{z_3 - z_1}\operatorname{dn} \lambda t.$$

343. The Rotation of the Line of Apsides.—The integration of z as a function of θ (Eq. (341.5)) is a more complicated integration,[1] and it will not be carried out here. Puiseux, however, has given an interesting demonstration of the fact that in making a complete oscillation the angle θ through which the pendulum turns is greater than 2π. Equation (341.5) can be written

$$\frac{dz}{d\theta} = \frac{a^2 - z^2}{a\sqrt{(a^2 - z_1^2)(a^2 - z_2^2)}} \times \sqrt{(z - z_1)(z_2 - z)([z_1 + z_2]z + [a^2 + z_1z_2])}. \tag{1}$$

Since, by Eq. (341.4),

$$z_3(z_1 + z_2) + (a^2 + z_1z_2) = 0,$$

the first term of which is negative and the second positive, it follows that, if $z < z_3$,

$$z(z_1 + z_2) + (a^2 + z_1z_2) > 0;$$

[1] See APPELL and LACOUR, "Théorie des Fonctions Elliptiques," p. 90.

and therefore, since

$$-a < z_1 < z < z_2 < +a,$$

in the motion of the pendulum,

$$-a(z_1 + z_2) + (a^2 + z_1z_2) > z(z_1 + z_2) + (a^2 + z_1z_2) > +a(z_1 + z_2) + (a^2 + z_1z_2). \quad (2)$$

But

$$(a^2 + z_1z_2) - a(z_1 + z_2) = (a - z_1)(a - z_2) = D^2,$$
$$(a^2 + z_1z_2) + a(z_1 + z_2) = (a + z_1)(a + z_2) = S^2,$$

the letters S and D being introduced for the sake of brevity. With this notation, the angle $\bar{\theta}$ through which the pendulum turns while it rises from z_1 to z_2, that is in a quarter period, is

$$\bar{\theta} = \int_{z_1}^{z_2} \frac{aSDdz}{(a^2 - z^2)\sqrt{(z - z_1)(z_2 - z)([z_1 + z_2]z + a^2 + z_1z_2)}}.$$

If it is borne in mind that

$$D > \sqrt{z(z_1 + z_2) + a^2 + z_1z_2} > S,$$

it is evident that

$$aS\int_{z_1}^{z_2} \frac{dz}{(a^2 - z^2)\sqrt{(z - z_1)(z_2 - z)}} < \bar{\theta} < aD\int_{z_1}^{z_2} \frac{dz}{(a^2 - z^2)\sqrt{(z - z_1)(z_2 - z)}}.$$

But

$$\int_{z_1}^{z_2} \frac{adz}{(a^2 - z^2)\sqrt{(z - z_1)(z_2 - z)}} = \frac{\pi}{2}\left(\frac{1}{S} + \frac{1}{D}\right),$$

and, therefore,

$$\frac{\pi}{2}\left(1 + \frac{S}{D}\right) < \bar{\theta} < \frac{\pi}{2}\left(1 + \frac{D}{S}\right). \quad (3)$$

Figure 172 represents the projection of the path of the pendulum upon a horizontal plane for the case

$$0 > z_2 > z_1.$$

The circle C_1 has a radius equal to $\sqrt{a^2 - z_1^2}$, and the circle C_2 has a radius equal to $\sqrt{a^2 - z_2^2}$. It will be observed that the projection of the path is something like a rotating ellipse, and that the direction of the rotation is the same as the direction of the motion of the bob

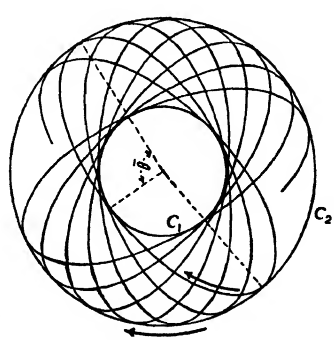


FIG. 172.

in its orbit. This is expressed more simply by saying that the rotation of the line of apsides is in the same direction as the motion of the pendulum in its orbit.

344. Special Cases.—*I. The Pendulum Rises Just to the Equator.* If the constant C is equal to h^2, it is shown by the last equation of Eq. (341.4) that the root z_2 is equal to zero; since z_1 is always negative, and z_3 is greater than $+a$. For this value of z_2, Eqs. (341.5) and (341.6) can be written

$$\left.\begin{aligned} d\theta &= \frac{a^2\sqrt{a^2 - z_1^2}\,dz}{(a^2 - z^2)\sqrt{z[z_1(a^2 - z^2) - z(a^2 - z_1^2)]}}, \\ \frac{\sqrt{2g}}{a}dt &= \frac{dz}{\sqrt{z[z_1(a^2 - z^2) - z(a^2 - z_1^2)]}}. \end{aligned}\right\} \quad (1)$$

Greenhill has shown that there exists a combination of these two expressions that can be integrated in terms of the elementary functions. Let the second equation of Eq. (1) be multiplied by $\sqrt{a^2 - z_1^2}\,/2$ and then subtracted from the first of Eq. (1). The result is

$$\begin{aligned} d\theta - \frac{\sqrt{2g(a^2 - z_1^2)}}{2a}dt &= \\ \frac{1}{2}\sqrt{a^2 - z_1^2}\,\frac{a^2 + z^2}{a^2 - z^2}&\,\frac{dz}{\sqrt{z[z_1(a^2 - z^2) - z(a^2 - z_1^2)}}. \end{aligned} \quad (2)$$

The equations of Eq. (341.4) become

$$z_1 + z_3 = \frac{h^2a^2}{2g}, \qquad z_1z_3 = -a^2,$$

from which it is found that

$$\frac{\sqrt{2g(a^2 - z_1^2)}}{2a} = \frac{1}{2}h\sqrt{-z_1}.$$

Hence, if the constant of integration is suitably determined, the integration of Eq. (2) gives

$$\theta - \frac{1}{2}h\sqrt{-z_1}\,t = \cos^{-1}\sqrt{\frac{z}{z_1}\cdot\frac{a^2 - z_1^2}{a^2 - z^2}}. \quad (3)$$

From Sec. 342, it is found that

$$z = z_1\mathrm{cn}^2\,\lambda t,$$

and, therefore, θ is completely defined as a function of the time.

II. The Roots z_1 and z_2 are Equal.—Since $z_1 + z_2$ must be negative, their common value is negative if the two are equal. On setting z_2 equal to z_1 in Eq. (341.6), it becomes

$$\frac{dz}{dt} = \frac{\sqrt{2g}}{a}\sqrt{(z - z_1)^2\left(z + \frac{a^2 + z_1^2}{2z_1}\right)}. \tag{1}$$

Since the factor

$$z + \frac{a^2 + z_1^2}{2z_1}$$

is negative for every value of z between $+a$ and $-a$, the only solution of Eq. (1) is

$$z = z_1;$$

that is, z is constant. Equation (340.3) then gives

$$\theta' = \frac{ha^2}{a^2 - z_1^2} = \sqrt{\frac{g}{-z_1}},$$

and, therefore,

$$\theta = \sqrt{\frac{g}{-z_1}}t.$$

The pendulum in this case describes a cone, and is therefore identical with the conical pendulum.

345. The Surface Reaction.—According to Sec. 339, the surface reaction is

$$R = -\left(f_r + m\frac{s'^2}{a}\right).$$

By Eq. (340.2),

$$\frac{ms'^2}{a} = -2mg \sin \varphi + maC,$$

and from Fig. 170,

$$f_r = -mg \sin \varphi.$$

Also (Eq. (341.4))

$$maC = 2mg\frac{(z_1 + z_2 + z_3)}{a} = 2mg\frac{z_1^2 + z_1z_2 + z_2^2 - a^2}{a(z_1 + z_2)},$$

so that

$$R = \frac{mg}{a}[3z - 2(z_1 + z_2 + z_3)]. \tag{1}$$

If R is negative the pendulum rod is under tension. The coordinate z varies from z_1 at the lowest point to z_2 at the highest point. Hence,

$$\frac{mg}{a}(z_1 - 2z_2 - 2z_3) \leqq R \leqq \frac{mg}{a}(-2z_1 + z_2 - 2z_3). \tag{2}$$

The tension changes to a thrust if z attains the value

$$z = \frac{2}{3}(z_1 + z_2 + z_3).$$

This can happen only if the last member of the inequality Eq. (2) is positive, for the left member is certainly negative; that is, if

$$z_2 > 2(z_1 + z_3).$$

Since $z_1 + z_3$ is positive, it is necessary that z_2 be positive.

If the tension in the rod changes to a thrust, the projection of the trajectory upon a horizontal plane has a point of inflection at the point where the change occurs; for the osculating plane of the trajectory, in which lies the resultant of all the forces which are acting upon the bob, at that instant passes through the vertical (the only force acting being gravity), and the radius of curvature changes sign.

Problems XXII

1. Taking the value of g as $32.173 - 0.085 \cos 2l$, where l is the latitude compute the length of the pendulum which makes one complete oscillation in 1 sec. for latitudes 35°, 40°, and 45°. *Ans.* 0.81425 ft.; 0.81460 ft.; 0.81495 ft.

2. The bob of a pendulum, which should beat once per second (that is, one complete oscillation) can be raised or lowered by means of a screw which has 50 threads to the inch. If the clock, to which the pendulum is attached, loses 1 min. per day, how many turns of the screw will make the clock keep time correctly? *Ans.* 0.679 = 245°.

3. If a pendulum which beats seconds in latitude 35° be carried to latitude 40°, how much will it gain in 24 hr.? *Ans.* 19 sec.

4. A particle, constrained to move on a circle, is attracted toward a fixed center on the circumference. What is the law of force if the reaction of the circle is constant?

5. Find the time required for a bead to slide down the curve

$$\frac{x^2}{a^2} = \frac{4}{9}\left(\frac{z}{a} - 1\right)^3$$

from $z = b$ to $z = c$, assuming that the speed vanishes for $z = 0$.

Ans. $T = (c - b)/\sqrt{2ga}$.

6. A bead slides freely on a smooth elliptic wire in a vertical plane, the equations of the ellipse being

$$x = a \cos \lambda, \qquad z = b \sin \lambda.$$

Show that

$$t = \frac{1}{\sqrt{2gb}} \int_{\lambda_0}^{\lambda} \sqrt{\frac{a^2 \sin^2 \lambda + b^2 \cos^2 \lambda}{\sin \lambda_0 - \sin \lambda}}\, d\lambda.$$

7. The particle A is constrained to move on the x-axis, and B is constrained to move on the y-axis. If they are at rest initially and attract each other according to any law which depends only on their mutual distance, show that they will arrive at the origin simultaneously.

8. A heavy bead slides along a curve in a vertical plane which is of such a nature that the vertical component of the velocity is constant. What is the equation of the curve? *Ans.* $(3gcx)^2 = (h - c^2 - 2gz)^3$.

9. A bead slides on a vertical plane curve under the action of gravity. What is the equation of the curve if its normal reaction is equal to n times the normal component of the weight of the bead? *Ans.* $n = 1$, a straight line; $n = 2$, a cycloid; $\cdots$.

10. Through the fixed point O_1 in space there pass n straight lines along which n particles slide. The particles start from O_1 at rest under the action of a force which is proportional to the distance toward a fixed point O_2. Show that the particles cross the sphere which has O_1O_2 as a diameter at the same instant.

11. A heavy particle, starting from rest at a point O, slides along a certain plane vertical curve and arrives at any point P of the curve in the same time that would have been required in sliding along the chord OP. What is the curve? [Euler.] *Ans.* A lemniscate.

12. Show that the lemniscate possesses the same property as in the preceding example when the weight of the particle is replaced by an attractive force with its center at O, the law of attraction being the direct first power. [Bonnet.]

13. A bead slides down a parabola, parameter p, which is in a vertical plane with its axis horizontal. The bead starts from rest at a height h above the axis. At what point does the reaction of the curve change sign? *Ans.* The point whose ordinate is the positive root of the equation

$$y^3 + 3p^2y - 2p^2h = 0.$$

14. The depth of a smooth spherical bowl is one and one-half times the radius of the sphere. A particle is projected horizontally from the bottom of the bowl with just enough speed to make it rise to the edge of the bowl, describe a parabola, and fall back to the opposite edge of the bowl. If a is the radius of the sphere, show that the speed at the top of the parabola is $\sqrt{ag/2}$, and that the speeds at the top of the parabola, the edge of the bowl, and the bottom of the bowl are in the ratios $1:2:\sqrt{10}$.

15. The cusps of a series of cycloids (Eq. (325.6)) are cut off by a line $y = -a$, leaving a series of detached cycloidal arcs. A particle is started into motion on one of the arcs, which are smooth, and arrives at the end of it with just sufficient speed to jump over to the next arc, which it follows to the end, then jumps the next gap, and so on. Show that its period is

$$T = \sqrt{\frac{a}{g}}\left[4\sin^{-1}\sqrt{\frac{2}{\pi}} + \sqrt{2\pi - 4}\right].$$

16. Find the plane tautochrone for an attractive force which is proportional to the distance, the origin being the point of tautochronism. *Ans.* $\theta = \text{const.}$; or $r = r_0e^{c\theta}$.

17. Show that if the oscillations of a spherical pendulum are infinitely small, the trajectory is an ellipse and the period is

$$T = 2\pi\sqrt{\frac{a}{g}}.$$

18. What is the period of a conical pendulum of length 4 ft. when the rod makes an angle of 30° with the vertical, and what is the tension in the rod if the bob weighs 5 lb.? *Ans.* 2.06 sec.; 5.77 lb.

19. A conical pendulum makes 40 revolutions per minute about the vertical axis, and the bob weighs 1 1/2 lb. Find the inclination of the rod to the vertical, and the tension in the rod. *Ans.* 23° 25′; 1.63 lb. $l = 2$ ft.

20. A particle moves on the surface of a sphere subject to a force which varies inversely as the cube of the distance from the xy-plane. Show that the trajectory is a spherical conic (the intersection of a sphere and a cone).

21. If a particle moves on a surface subject to no forces except the constraint of the surface and friction, the trajectory is a geodesic.

22. Show that the tension in the rod of a spherical pendulum cannot change into a thrust unless $\cos\varphi_1 < 1/3$ ($z_1 = a\sin\varphi_1$); and if this condition is satisfied the change will occur if, and only if,

$$(2z_2 + z_1)^2 < 9z_1^2 - 8a^2.$$

23. A particle moves upon the helicoidal surface

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad z = k\theta,$$

under the action of a force $m\mu r$ directed away from the z-axis. Show that the motion of the particle is defined by the two independent equations

$$r'^2 + (r^2 + k^2)\theta'^2 - \mu r^2 = c,$$
$$(r^2 + k^2)\theta' = h.$$

24. A family of cycloids has the same base and a cusp at O in common, but the radii of the rolling circles are different. If particles slide down the various cycloids under the action of gravity, starting simultaneously from rest at the point O, show that at any instant the curve formed by the particles (the synchronous curve) is an orthogonal curve of the family of cycloids. [Euler.]

CHAPTER XIV

THE GENERALIZED COORDINATES OF LAGRANGE

I. TRANSFORMATION OF COORDINATES

346. Changing the System of Coordinates.—It is frequently desirable, or even necessary, to change the variables from one system to another, a process with which the student is already familiar. It was seen in the last chapter that the equations of constrained motion of a particle in rectangular coordinates, three in number, are not independent. By changing to spherical coordinates, it was found that the equations of motion for a particle constrained to the surface of a sphere were reduced in number to two, and that these two equations were independent. For motion on a given surface, two coordinates are sufficient to define the position of the particle; and therefore, in general, two equations, each of the second order, are sufficient to define the motion. For motion along a curve, one coordinate is sufficient to define the position of the particle; and one equation of the second order is sufficient to define the motion.

Lagrange has shown in a very general manner how this change of variables can be effected, whether the change be accompanied by a reduction in the number of variables or not. Indeed, it is a relatively simple matter to write down the equations of motion, *ab initio,* in any desired system of coördinates by means of the form which Lagrange gave to the equations of motion.

347. The Equations of Lagrange for a Free Particle.—Expressed in rectangular coordinates, the equations of motion of a free particle are

$$mx'' = X, \qquad my'' = Y, \qquad mz'' = Z; \tag{1}$$

where X, Y, and Z are functions of x, y, z; x', y', z', and represent the components of the forces acting along the x-, y-, and z-axes, respectively. It will be supposed that the new variables are q_1,

q_2, and q_3, and that they are related to the variables x, y, and z by the equations

$$\left.\begin{aligned} x &= \varphi_1(q_1, q_2, q_3; t), \\ y &= \varphi_2(q_1, q_2, q_3; t), \\ z &= \varphi_3(q_1, q_2, q_3; t); \end{aligned}\right\} \tag{2}$$

so that the equations of transformation (Eq. (2)) may contain the time explicitly, or they may not: it is immaterial. In the following analysis, it will be understood that

$$\frac{\partial x}{\partial q_i} \equiv \frac{\partial \varphi_1}{\partial q_i}, \qquad \frac{\partial y}{\partial q_i} \equiv \frac{\partial \varphi_2}{\partial q_i}, \qquad \frac{\partial z}{\partial q_i} \equiv \frac{\partial \varphi_3}{\partial q_i}.$$

If the first equation of Eq. (1) is multiplied by $\partial x/\partial q_1$, the second by $\partial y/\partial q_1$, and the third by $\partial z/\partial q_1$, and the three equations are then added, the result is

$$m\left(\frac{\partial x}{\partial q_1}x'' + \frac{\partial y}{\partial q_1}y'' + \frac{\partial z}{\partial q_1}z''\right) = X\frac{\partial x}{\partial q_1} + Y\frac{\partial y}{\partial q_1} + Z\frac{\partial z}{\partial q_1}. \tag{3}$$

For brevity of notation, let

$$Q_i = X\frac{\partial x}{\partial q_i} + Y\frac{\partial y}{\partial q_i} + Z\frac{\partial z}{\partial q_i}, \qquad i = 1, 2, 3.$$

It will be verified then without much difficulty that Eq. (3) can be written

$$\begin{aligned} &\frac{d}{dt}\left[m\left(\frac{\partial x}{\partial q_1}x' + \frac{\partial y}{\partial q_1}y' + \frac{\partial z}{\partial q_1}z'\right)\right] \\ &\qquad -m\left[x'\left(\frac{\partial x}{\partial q_1}\right)' + y'\left(\frac{\partial y}{\partial q_1}\right)' + z'\left(\frac{\partial z}{\partial q_1}\right)'\right] = Q_1, \end{aligned} \tag{4}$$

where, as usual

$$\left(\frac{\partial x}{\partial q_1}\right)' \equiv \frac{d}{dt}\left(\frac{\partial x}{\partial q_1}\right), \qquad \text{etc.}$$

On differentiating the equations of transformation (Eq. (2)) with respect to the time, the following equations result:

$$\left.\begin{aligned} x' &= \frac{\partial x}{\partial q_1}q_1' + \frac{\partial x}{\partial q_2}q_2' + \frac{\partial x}{\partial q_3}q_3' + \frac{\partial x}{\partial t}, \\ y' &= \frac{\partial y}{\partial q_1}q_1' + \frac{\partial y}{\partial q_2}q_2' + \frac{\partial y}{\partial q_3}q_3' + \frac{\partial y}{\partial t}, \\ z' &= \frac{\partial z}{\partial q_1}q_1' + \frac{\partial z}{\partial q_2}q_2' + \frac{\partial z}{\partial q_3}q_3' + \frac{\partial z}{\partial t}. \end{aligned}\right\} \tag{5}$$

348. If There Exists a Force Function.—The quantities which have been denoted by Q_i are defined by the equations

$$Q_i = X\frac{\partial x}{\partial q_i} + Y\frac{\partial y}{\partial q_i} + Z\frac{\partial z}{\partial q_i}.$$

If there exists a force function $U(x, y, z; t)$ such that

$$X = \frac{\partial U}{\partial x}, \qquad Y = \frac{\partial U}{\partial y}, \qquad Z = \frac{\partial U}{\partial z},$$

then

$$Q_i = \frac{\partial U}{\partial x}\frac{\partial x}{\partial q_i} + \frac{\partial U}{\partial y}\frac{\partial y}{\partial q_i} + \frac{\partial U}{\partial z}\frac{\partial z}{\partial q_i};$$

and since U is a function of the q_i only as these letters enter through x, y, and z, not t, this expression is the derivative of U with respect to q_i; that is,

$$Q_i = \frac{\partial U}{\partial q_i}.$$

The equations of Lagrange then become

$$\left(\frac{\partial T}{\partial q_i'}\right)' - \frac{\partial T}{\partial q_i} = \frac{\partial U}{\partial q_i} \qquad i = 1, 2, 3. \tag{1}$$

The force function U is the negative of the potential energy, that is,

$$U = -V.$$

Form the difference between the kinetic and potential energies, and let

$$L = T + U = T - V.$$

Since U does not contain q_1', q_2', q_3', it is evident that

$$\frac{\partial L}{\partial q_i'} = \frac{\partial T}{\partial q_i'}, \qquad \frac{\partial L}{\partial q_i} = \frac{\partial T}{\partial q_i} + \frac{\partial U}{\partial q_i};$$

therefore, Lagrange's equations become simply

$$\left(\frac{\partial L}{\partial q_i'}\right)' - \frac{\partial L}{\partial q_i} = 0 \qquad i = 1, 2, 3. \tag{2}$$

The function L is called the *Lagrangean function*, and its negative was called by Helmholtz the *kinetic potential.*

349. The Energy and Other Integrals.—It was proved in Sec. 269 that whenever there exists a force function in terms of the variables x, y, and z which is independent of the time, there also exists an energy integral. Evidently, the energy integral, when

it exists, will be transformed by a direct substitution of the equations of transformation (347.2) into an integral in the new variables q_1, q_2, and q_3 and their derivatives; but if the equations of transformation contain the time explicitly, the integral, expressed in the new variables, also, in general, will contain the time explicitly. If the equations of transformation do not contain the time explicitly, of course, the transformed integral will not contain the time explicitly.

It will be supposed, for simplicity, that the equations of transformations do not contain the time explicitly. Then

$$\left.\begin{aligned} x' &= \frac{\partial x}{\partial q_1}q_1' + \frac{\partial x}{\partial q_2}q_2' + \frac{\partial x}{\partial q_3}q_3', \\ y' &= \frac{\partial y}{\partial q_1}q_1' + \frac{\partial y}{\partial q_2}q_2' + \frac{\partial y}{\partial q_3}q_3', \\ z' &= \frac{\partial z}{\partial q_1}q_1' + \frac{\partial z}{\partial q_2}q_2' + \frac{\partial z}{\partial q_3}q_3'; \end{aligned}\right\} \tag{1}$$

and

$$T = \frac{1}{2}m(x'^2 + y'^2 + z'^2),$$

when expressed in the new variables, will be a homogeneous quadratic form in q_1', q_2', q_3'.

Let Eq. (348.1) be multiplied by q_i', and then summed as to i. The result is

$$\sum_{i=1}^{3} q_i'\left(\frac{\partial T}{\partial q_i'}\right)' - \sum_{i=1}^{3} \frac{\partial T}{\partial q_i}q_i' = \sum_{i=1}^{3} \frac{\partial U}{\partial q_i}q_i'.$$

Now

$$q_i'\left(\frac{\partial T}{\partial q_i'}\right)' = \left(q_i'\frac{\partial T}{\partial q_i'}\right)' - \frac{\partial T}{\partial q_i'}q_i''.$$

Hence,

$$\sum_{i=1}^{3}\left(q_i'\frac{\partial T}{\partial q_i'}\right)' - \sum_{i=1}^{3}\frac{\partial T}{\partial q_i'}q_i'' - \sum_{i=1}^{3}\frac{\partial T}{\partial q_i}q_i' = \sum_{i=1}^{3}\frac{\partial U}{\partial q_i}q_i'. \tag{2}$$

If the potential function $U(x, y, z)$ does not contain the time explicitly, then $U(q_1, q_2, q_3)$ does not contain it and

$$\sum_{i=1}^{3}\frac{\partial U}{\partial q_i}q_i' = U'.$$

Also, since T is a function of q_i and q_i',

$$T' = \sum \frac{\partial T}{\partial q_i'}q_i'' + \sum \frac{\partial T}{\partial q_i}q_i'; \tag{3}$$

and by Euler's theorem on homogeneous functions

$$q_1'\frac{\partial T}{\partial q_1'} + q_2'\frac{\partial T}{\partial q_2'} + q_3'\frac{\partial T}{\partial q_3'} = 2T; \tag{4}$$

so that Eq. (2) becomes

$$2T' - T' = U'.$$

Therefore,

$$T = U + C, \tag{5}$$

which is the energy integral.

It may happen, however, that the potential function, when expressed in the rectangular coordinates, contains the time explicitly, but that one can find a transformation (Eq. (347.2)) which not only frees the potential function of the time explicitly, but also leaves the expression for the kinetic energy free from the time explicitly. In this event Eqs. (2) and (3) remain unaltered, but Eq. (1) takes the non-homogeneous form of Eq. (347.5); and therefore T, the kinetic energy, is non-homogeneous in q_1', q_2', q_3'. However, let

$$T = T_2 + T_1 + T_0,$$

where T_2 is the sum of the terms which are homogeneous of the second degree in q_1', q_2', q_3', and T_1 and T_0 are homogeneous of degrees 1 and 0, respectively. Then Eq. (4) takes the form

$$\sum_{i=1}^{3} q_i'\frac{\partial T}{\partial q_i'} = 2T_2 + T_1,$$

and

$$T' = T_2' + T_1' + T_0'.$$

Equation (2) now becomes

$$T_2' - T_0' = U';$$

and, therefore,

$$T_2 - T_0 = U + C, \tag{6}$$

an integral which is analogous to the energy integral. The inverse substitution will, of course, give the corresponding integral in terms of the rectangular coordinates and the time.

If one of the coordinates, q_3 for example, does not occur in the expression for the Lagrangean function L, the corresponding differential equation reduces to

$$\left(\frac{\partial L}{\partial q_3'}\right)' = 0;$$

and, therefore,

$$\frac{\partial L}{\partial q_3'} = \text{constant}$$

is an integral of the differential equations. This is what happens for the integrals of areas. The coordinate q_3, under these circumstances, is called a *cyclic* or *ignorable coordinate.* For each ignorable coordinate there exists an integral.

350. A Vector Interpretation of the Equation of Lagrange.—It will be helpful to an understanding of the equations of Lagrange to examine the nature of these equations. The equations of transformation (Eq. (2)) are

$$x = \varphi_1(q_1, q_2, q_3; t), \qquad y = \varphi_2(q_1, q_2, q_3; t), \qquad z = \varphi_3(q_1, q_2, q_3; t)$$

If q_2, q_3, and t are kept fixed but q_1 is varied, the point x, y, z undergoes a displacement of which the components are

$$dx = \frac{\partial x}{\partial q_1}dq_1, \qquad dy = \frac{\partial y}{\partial q_1}dq_1, \qquad dz = \frac{\partial z}{\partial q_1}dq_1;$$

and the magnitude of the displacement is

$$ds = \sqrt{\left(\frac{\partial x}{\partial q_1}\right)^2 + \left(\frac{\partial y}{\partial q_1}\right)^2 + \left(\frac{\partial z}{\partial q_1}\right)^2}dq_1 = R_1 dq_1,$$

where

$$R_1 = \sqrt{\left(\frac{\partial x}{\partial q_1}\right)^2 + \left(\frac{\partial y}{\partial q_1}\right)^2 + \left(\frac{\partial z}{\partial q_1}\right)^2}.$$

Hence,

$$\frac{dx}{ds} = \frac{1}{R_1}\frac{\partial x}{\partial q_1}, \qquad \frac{dy}{ds} = \frac{1}{R_1}\frac{\partial y}{\partial q_1}, \qquad \frac{dz}{ds} = \frac{1}{R_1}\frac{\partial z}{\partial q_1};$$

and since $\frac{dx}{ds}$, $\frac{dy}{ds}$, and $\frac{dz}{ds}$ are the direction cosines of the displacement vector, the expressions

$$\frac{1}{R_1}\frac{\partial x}{\partial q_1}, \qquad \frac{1}{R_1}\frac{\partial y}{dq_1}, \qquad \frac{1}{R_1}\frac{\partial z}{\partial q_1} \tag{1}$$

also are the direction cosines of the displacement vector due to a change in q_1; or, for short, the direction cosines of the q_1-direction.

Let **A** be the acceleration vector of which the components are x'', y'', and z''. Then

$$\frac{1}{R_1}\left(\frac{\partial x}{\partial q_1}x'' + \frac{\partial y}{\partial q_1}y'' + \frac{\partial z}{\partial q_1}z''\right)$$

is the component of the acceleration in the q_1-direction. The change from the left member of Eq. (347.3) to the left member of the first equation of Eq. (347.8) is merely a change of form of expression, the mechanical significance remaining unaltered. Hence, if $\mathbf{A}_{q_1}$ is the component of **A** in the q_1-direction,

$$\frac{1}{R_1}\left[\left(\frac{\partial T}{\partial q_1'}\right)' - \frac{\partial T}{\partial q_1}\right] \equiv mA_{q_1}.$$

Likewise, if **F** is the force which is acting, with the components **X**, **Y**, and **Z** along the x-, y-, and z-axes, and if $\mathbf{F}_{q_1}$ is the component of the force in the q_1-direction, then it is evident that

$$\frac{1}{R}\left(X\frac{\partial x}{\partial q_1} + Y\frac{\partial y}{\partial q_1} + Z\frac{\partial z}{\partial q_1}\right) = F_{q_1} = \frac{Q_1}{R_1}.$$

Hence, by Newton's second law,

$$\frac{1}{R_1}\left[\left(\frac{\partial T}{\partial q_1'}\right)' - \frac{\partial T}{\partial q_1}\right] = \frac{Q_1}{R_1},$$

which, aside from the factor $1/R_1$, is the same as the first equation of Eq. (8). The interpretation of the other two equations is similar.

351. Transformation to Polar Coordinates.—Suppose that it is desired to change from rectangular coordinates to polar coordinates in the plane. The equations of transformation are

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad z = 0.$$

Let r play the rôle of q_1 and θ that of q_2. Then

$$\frac{\partial x}{\partial r} = \cos\theta, \qquad \frac{\partial y}{\partial r} = \sin\theta, \qquad R_r = \sqrt{\cos^2\theta + \sin^2\theta} = 1;$$

$$\frac{\partial x}{\partial \theta} = -r\sin\theta, \quad \frac{\partial y}{\partial \theta} = r\cos\theta, \quad R_\theta = \sqrt{r^2\sin^2\theta + r^2\cos^2\theta} = r.$$

The arc element ds in polar coordinates is

$$ds^2 = dr^2 + r^2 d\theta^2,$$

so that

$$T = \frac{1}{2}m(r'^2 + r^2\theta'^2).$$

The r-direction is evidently along the radius vector away from the origin, and the θ-direction is perpendicular to the r-direction, directed 90° ahead. Hence, the components of the acceleration in these directions are

$$\alpha_r = \frac{1}{R_r}\left\{\left[\frac{\partial}{\partial r'}\left(\frac{1}{2}r'^2 + \frac{1}{2}r^2\theta'^2\right)\right]' - \frac{1}{2}\frac{\partial}{\partial r}\left[r'^2 + r^2\theta'^2\right]\right\} = r'' - r\theta'^2,$$

$$\alpha_\theta = \frac{1}{R_\theta}\left\{\left[\frac{\partial}{\partial \theta'}\left(\frac{1}{2}r'^2 + \frac{1}{2}r'\theta'^2\right)\right]' - \frac{1}{2}\frac{\partial}{\partial \theta}\left[r'^2 + r^2\theta'^2\right]\right\} = r\theta'' + 2r'\theta',$$

which are the same expressions that were derived in Sec. 257.

352. Spherical Coordinates.—The equations of transformation are

$$x = r\cos\varphi\cos\theta, \qquad y = r\cos\varphi\sin\theta, \qquad z = r\sin\varphi.$$

From these equations, the following relations are derived:

$$\frac{\partial x}{\partial r} = +\ \cos\varphi\cos\theta, \qquad \frac{\partial y}{\partial r} = +\ \cos\varphi\sin\theta, \qquad \frac{\partial z}{\partial r} = \sin\varphi,$$

$$\frac{\partial x}{\partial \varphi} = -r\sin\varphi\cos\theta, \qquad \frac{\partial y}{\partial \varphi} = -r\sin\varphi\sin\theta, \qquad \frac{\partial z}{\partial \varphi} = r\cos\varphi,$$

$$\frac{\partial x}{\partial \theta} = -r\cos\varphi\sin\theta, \qquad \frac{\partial y}{\partial \theta} = +r\cos\varphi\cos\theta, \qquad \frac{\partial z}{\partial \theta} = 0.$$

Hence,

$$R_r = 1, \qquad R_\varphi = r, \qquad R_\theta = r\cos\varphi.$$

The arc element, in spherical coordinates, is

$$ds^2 = dr^2 + r^2 d\varphi^2 + r^2\cos^2\varphi\, d\theta^2;$$

so that

$$s'^2 = r'^2 + r^2\varphi'^2 + r^2\cos^2\varphi\, \theta'^2.$$

The r-direction is along the radius vector directed outward; the φ-direction is along a meridian directed toward the north pole; the θ-direction is along a parallel of latitude in the direction of increasing longitude. Hence,

$$\alpha_r = \frac{1}{2R_r}\left[\left(\frac{\partial s'^2}{\partial r'}\right)' - \frac{\partial s'^2}{\partial r}\right] = r'' - r\varphi'^2 - r\cos^2\varphi\cdot\theta'^2,$$

$$\alpha_\varphi = \frac{1}{2R_\varphi}\left[\left(\frac{\partial s'^2}{\partial \varphi'}\right)' - \frac{\partial s'^2}{\partial \varphi}\right] = r\varphi'' + 2r'\varphi' + r\sin\varphi\cos\varphi\cdot\theta'^2,$$

$$\alpha_\theta = \frac{1}{2R_\theta}\left[\left(\frac{\partial s'^2}{\partial \theta'}\right)' - \frac{\partial s'^2}{\partial \theta}\right] = r\cos\varphi\cdot\theta'' + 2\cos\varphi\cdot r'\theta' - 2r\sin\varphi\cdot\varphi'\theta',$$

which are the same as those derived in Sec. 263; but they were derived with much less labor.

353. Elliptic Coordinates.—If q is regarded as a parameter, the equation

$$\frac{x^2}{a^2 - q} + \frac{y^2}{b^2 - q} + \frac{z^2}{c^2 - q} - 1 = 0, \qquad a^2 > b^2 > c^2, \tag{1}$$

represents a family of *confocal conicoids*, the type of the conicoid depending upon the value of q, as follows:

$$\left.\begin{array}{ll} q < c^2 & \text{(ellipsoid)}, \\ c^2 < q < b^2 & \text{(hyperboloid of one sheet)}, \\ b^2 < q < a^2 & \text{(hyperboloid of two sheets)}, \\ a^2 < q & \text{(imaginary surface)}. \end{array}\right\} \tag{2}$$

Through any fixed point x, y, z of space, there passes one and only one conicoid of each type. In order to determine the corresponding value of the parameter q, let the coordinates of the given point be substituted in Eq. (1), and then solve the equation for q. When Eq. (1) is cleared of fractions, it is seen that it is a cubic in q, and that there are accordingly three roots. In order to show that the three roots are real, let the left member of Eq. (1) be denoted by $f(q)$. The derivative of $f(q)$ is always positive, for

$$\frac{df}{dq} = \frac{x^2}{(a^2 - q)^2} + \frac{y^2}{(b^2 - q)^2} + \frac{z^2}{(c^2 - q)^2}. \tag{3}$$

A Triply Orthogonal System of Surfaces.—The correspondence between the values of q and the values of f is as follows:

$$\begin{array}{lcccccccc} q = & -\infty, & c^2 - 0, & c^2 + 0, & b^2 - 0, & b^2 + 0, & a^2 - 0, & a^2 + 0, & +\infty; \\ f = & -1, & +\infty, & -\infty, & +\infty, & -\infty, & +\infty, & -\infty, & -1. \end{array}$$

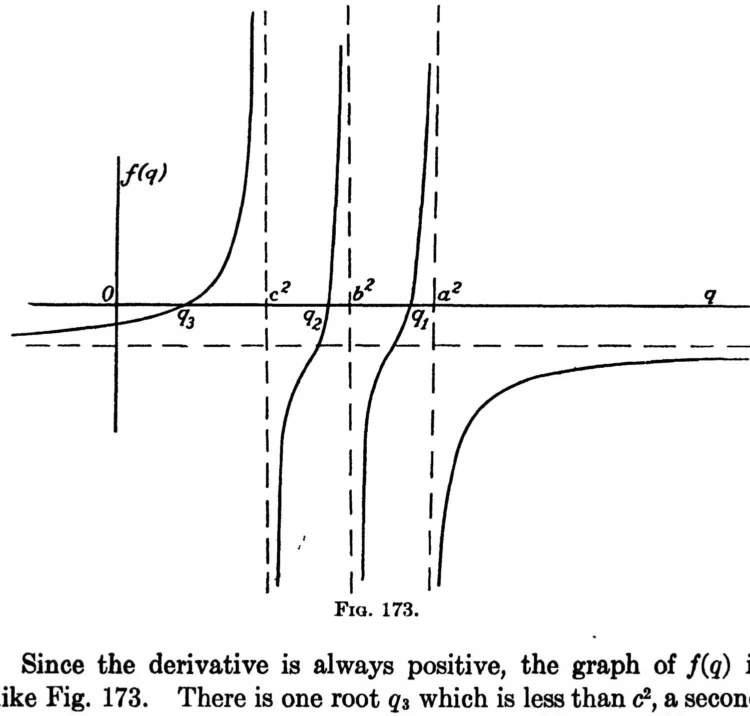


FIG. 173.

Since the derivative is always positive, the graph of $f(q)$ is like Fig. 173. There is one root q_3 which is less than c^2, a second one q_2 lies between c^2 and b^2, and the third q_1 lies between b^2 and a^2. Therefore, by Eq. (2),

$$
\begin{aligned}
&\frac{x^2}{a^2 - q_1} + \frac{y^2}{b^2 - q_1} + \frac{z^2}{c^2 - q_1} = 1 \qquad \text{(hyp. of two sheets)},\\
&\frac{x^2}{a^2 - q_2} + \frac{y^2}{b^2 - q_2} + \frac{z^2}{c^2 - q_2} = 1 \qquad \text{(hyp. of one sheet)},\\
&\frac{x^2}{a^2 - q_3} + \frac{y^2}{b^2 - q_3} + \frac{z^2}{c^2 - q_3} = 1 \qquad \text{(ellipsoid)}.
\end{aligned} \tag{4}
$$

The direction cosines α, β, and γ of the normals to these three surfaces are

$$
\left.
\begin{aligned}
\alpha_1 &= \frac{1}{R_1^{(0)}}\frac{x}{a^2 - q_1}, & \beta_1 &= \frac{1}{R_1^{(0)}}\frac{y}{b^2 - q_1}, & \gamma_1 &= \frac{1}{R_1^{(0)}}\frac{z}{c^2 - q_1};\\
\alpha_2 &= \frac{1}{R_2^{(0)}}\frac{x}{a^2 - q_2}, & \beta_2 &= \frac{1}{R_2^{(0)}}\frac{y}{b^2 - q_2}, & \gamma_2 &= \frac{1}{R_2^{(0)}}\frac{z}{c^2 - q_2};\\
\alpha_3 &= \frac{1}{R_3^{(0)}}\frac{x}{a^2 - q_3}, & \beta_3 &= \frac{1}{R_3^{(0)}}\frac{y}{b^2 - q_3}, & \gamma_3 &= \frac{1}{R_3^{(0)}}\frac{z}{c^2 - q_3}.
\end{aligned}
\right\} \tag{5}
$$

Hence, the cosine of the angle between the normals to the hyperboloid of two sheets and the hyperboloid of one sheet is

$$
\begin{aligned}
\alpha_1\alpha_2 + \beta_1\beta_2 + \gamma_1\gamma_2 = \frac{1}{R_1^{(0)}R_2^{(0)}}\bigg[&\frac{x^2}{(a^2 - q_1)(a^2 - q_2)} + \\
&\frac{y^2}{(b^2 - q_1)(b^2 - q_2)} + \frac{z^2}{(c^2 - q_1)(c^2 - q_2)}\bigg].
\end{aligned} \tag{6}
$$

If the second equation of Eq. (4) be subtracted from the first, it is found that

$$
\begin{aligned}
(q_1 - q_2)\bigg[&\frac{x^2}{(a^2 - q_1)(a^2 - q_2)} + \frac{y^2}{(b^2 - q_1)(b^2 - q_2)} + \\
&\frac{z^2}{(c^2 - q_1)(c^2 - q_2)}\bigg] = 0.
\end{aligned} \tag{7}
$$

Hence,

$$\alpha_1\alpha_2 + \beta_1\beta_2 + \gamma_1\gamma_2 = 0;$$

and likewise,

$$
\begin{aligned}
\alpha_2\alpha_3 + \beta_2\beta_3 + \gamma_2\gamma_3 &= 0,\\
\alpha_3\alpha_1 + \beta_3\beta_1 + \gamma_3\gamma_1 &= 0.
\end{aligned}
$$

Thus, the three surfaces intersect one another always at right angles, and they are said to form a *triply orthogonal system.* The three roots q_1, q_2, and q_3 are called the *elliptic coordinates* of the point x, y, z. Their order of magnitude is

$$a^2 > q_1 > b^2 > q_2 > c^2 > q_3.$$

The Equations of Transformation.—On clearing Eq. (353.1) of fractions, it will be observed that

$$
\begin{aligned}
x^2(b^2 - q)(c^2 - q) + y^2(a^2 - q)(c^2 - q) + z^2(a^2 - q)(b^2 - q)&\\
- (a^2 - q)(b^2 - q)(c^2 - q) \equiv (q - q_1)(q - q_2)(q - q_3)&.
\end{aligned} \tag{8}
$$

This is merely an identity, and holds for every value of q; but if q is set equal to a^2, b^2, and c^2, successively, the equations of transformation (Eq. (347.2)) are obtained easily, namely,

$$\left.\begin{aligned} x^2 &= \frac{(a^2 - q_1)(a^2 - q_2)(a^2 - q_3)}{(a^2 - b^2)(a^2 - c^2)}, \\ y^2 &= \frac{(q_1 - b^2)(b^2 - q_2)(b^2 - q_3)}{(a^2 - b^2)(b^2 - c^2)}, \\ z^2 &= \frac{(q_1 - c^2)(q_2 - c^2)(c^2 - q_3)}{(a^2 - c^2)(b^2 - c^2)}. \end{aligned}\right\} \tag{9}$$

If q_3 is kept fixed while q_2 and q_1 vary, the point x, y, z describes the ellipsoid. Thus it can be said that q_3 equal to a constant is

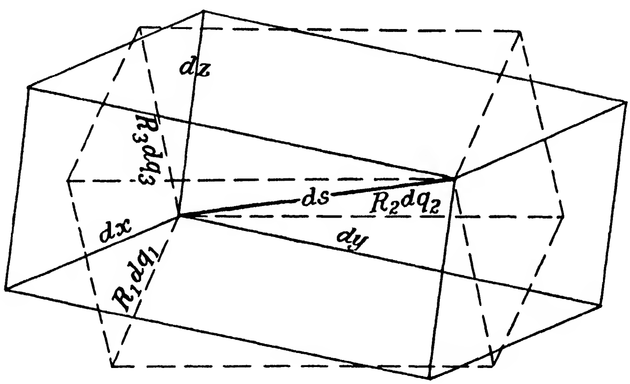


FIG. 174.

the ellipsoid, q_2 constant is the hyperboloid of one sheet, and q_1 constant is the hyperboloid of two sheets. It is readily verified from Eqs. (9) and (5) that the q_3-direction is normal to the ellipsoid in the sense q_3 increasing, the q_2-direction is normal to the hyperboloid of one sheet, and the q_1-direction is normal to the hyperboloid of two sheets. Since these directions are mutually at right angles, it follows that

$$ds^2 = R_1^2dq_1^2 + R_2^2dq_2^2 + R_3^2dq_3^2;$$

where R_1^2, R_2^2, and R_3^2 are certain functions of q_1, q_2, and q_3 which were defined in Sec. 350.

The Expression for the Kinetic Energy.—On differentiating Eq. (9) logarithmically, there results

$$dx = \frac{x}{2}\left[-\frac{dq_1}{a^2 - q_1} - \frac{dq_2}{a^2 - q_2} - \frac{dq_3}{a^2 - q_3}\right],$$
$$dy = \frac{y}{2}\left[-\frac{dq_1}{b^2 - q_1} - \frac{dq_2}{b^2 - q_2} - \frac{dq_3}{b^2 - q_3}\right],$$
$$dz = \frac{z}{2}\left[-\frac{dq_1}{c^2 - q_1} - \frac{dq_2}{c^2 - q_2} - \frac{dq_3}{c^2 - q_3}\right].$$

Squaring and adding, the cross-product terms disappear by virtue of relations in Eq. (7). The coefficient of dq_1^2 is

$$\left(\frac{\partial x}{\partial q_1}\right)^2 + \left(\frac{\partial y}{\partial q_1}\right)^2 + \left(\frac{\partial z}{\partial q_1}\right)^2 = \frac{1}{4}\left[\frac{x^2}{(a^2 - q_1)^2} + \frac{y^2}{(b^2 - q_1)^2} + \frac{z^2}{(c^2 - q_1)^2}\right];$$

that is,

$$R_1^2 = \frac{1}{4}\frac{\partial f}{\partial q}\bigg|_{q=q_1}, \text{ by Eq. (3).}$$

Now, from Eq. (8),

$$f(q) \equiv \frac{(q - q_1)(q - q_2)(q - q_3)}{(a^2 - q)(b^2 - q)(c^2 - q)}; \tag{10}$$

so that

$$\frac{df}{dq} = \frac{(q - q_2)(q - q_3)}{(a^2 - q)(b^2 - q)(c^2 - q)} + (q - q_1)\left[\ \cdots\ \right],$$

where the terms not computed carry $(q - q_1)$ as a factor, and therefore vanish when q is set equal to q_1. Therefore,

$$R_1^2 = \frac{1}{4}\frac{(q_2 - q_1)(q_3 - q_1)}{(a^2 - q_1)(b^2 - q_1)(c^2 - q_1)};$$

and, similarly,

$$R_2^2 = \frac{1}{4}\frac{(q_2 - q_1)(q_2 - q_3)}{(a^2 - q_2)(b^2 - q_2)(c^2 - q_2)},$$
$$R_3^2 = \frac{1}{4}\frac{(q_1 - q_3)(q_2 - q_3)}{(a^2 - q_3)(b^2 - q_3)(c^2 - q_3)}.$$

The expression for the kinetic energy is now easily written. It is

$$T = \frac{1}{2}m[R_1^2q_1'^2 + R_2^2q_2'^2 + R_3^2q_3'^2];$$

and from this expression the equation of motion can be derived without further difficulty.

If F_1, F_2, and F_3 are the components of the force which is acting in the q_1-, q_2-, and q_3-directions, then (Sec. 350)

$$Q_1 = F_1R_1, \qquad Q_2 = F_2R_2, \qquad Q_3 = F_3R_3.$$

Elliptic Coordinates in the Plane.—If q_1, q_2, and q_3 are regarded as the independent variables, and x, y, and z are determined by Eq. (9), it is seen that if q_3 approaches c^2, the value of z^2 tends toward zero; in the limit, for $q_3 = c^2$, z is zero, and

$$x^2 = \frac{(a^2 - q_1)(a^2 - q_2)}{(a^2 - b^2)}, \qquad y^2 = \frac{(q_1 - b^2)(b^2 - q_2)}{(a^2 - b^2)}.$$

The limits of the three surfaces (Eq. (4)) are

$$\frac{x^2}{a^2 - q_1} + \frac{y^2}{b^2 - q_1} = 1 \qquad \text{(hyperbola)},$$

$$\frac{x^2}{a^2 - q_2} + \frac{y^2}{b^2 - q_2} = 1 \qquad \text{(ellipse)},$$

and the third is arbitrary.

Since z and q_3 are constants, the expression for the arc element becomes

$$ds^2 = R_1^2 dq_1^2 + R_2^2 dq_2^2,$$

with

$$R_1^2 = \frac{1}{4}\frac{q_2 - q_1}{(a^2 - q_1)(b^2 - q_1)}, \qquad R_2^2 = \frac{1}{4}\frac{q_2 - q_1}{(a^2 - q_2)(b^2 - q_2)}.$$

The q_1-direction is normal to the hyperbola, and the q_2-direction is normal to the ellipse. If the force is resolved into its two components F_1 and F_2 in these directions, then

$$Q_1 = F_1 R_1, \qquad Q_2 = F_2 R_2.$$

These relations, together with the expression for the kinetic energy,

$$T = \frac{1}{2}m(R_1^2 q_1'^2 + R_2^2 q_2'^2),$$

are sufficient for writing down the equations of motion.

II. MOVING CONSTRAINTS

354. Moving Constraints.—If a particle is constrained to move along a fixed curve, the equation of the curve can be expressed parametrically. Thus,

$$x = \varphi_1(q), \qquad y = \varphi_2(q), \qquad z = \varphi_3(q);$$

but if the curve itself has a prescribed motion, its parametric equation has the form

$$x = \varphi_1(q, t), \qquad y = \varphi_2(q, t), \qquad z = \varphi_3(q, t).$$

Similarly, if a particle is constrained to move on a fixed surface, the coordinates can be expressed in terms of two parameters

$$x = \varphi_1(q_1, q_2), \qquad y = \varphi_2(q_1, q_2), \qquad z = \varphi_3(q_1, q_2).$$

If the surface is in motion, in some prescribed manner, the expressions for the coordinates x, y, and z have the form

$$x = \varphi_1(q_1, q_2; t), \qquad y = \varphi_2(q_1, q_2; t), \qquad z = \varphi_3(q_1, q_2; t).$$

All four of the above cases are particular instances of the transformation of Lagrange (Eq. (347.2)) in which the number of parameters is less than the original number of the coordinates, the number of the parameters being equal to the number of the degrees of freedom of the particle. In all such cases the equations of Lagrange are immediately applicable. Two illustrations of the application of these equations will be given.

355. The Particle is Constrained to a Moving Line.—A heavy bead is free to slide without friction along a straight line which describes a right circular cone, about a vertical axis, with uniform angular speed ω. It is required to describe the motion of the bead.

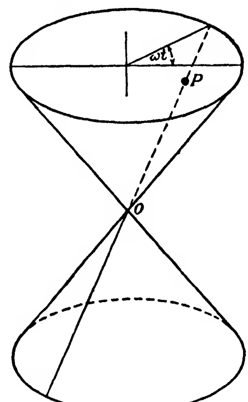


FIG. 175.

Let r be the distance of the bead from the apex of the cone, positive if above the vertex and negative if below the vertex. Let α be the angle which the line makes with the vertical axis. Then the rectangular coordinates of the bead are (Fig. 175)

$$x = r \sin \alpha \cos \omega t, \quad y = r \sin \alpha \sin \omega t, \quad z = r \cos \alpha, \tag{1}$$

in which r plays the rôle of the parameter q, and α and ω are constants.

The derivatives with respect to the time can be written

$$x' = \frac{x}{r} r' - \omega y, \qquad y' = \frac{y}{r} r' + \omega x, \qquad z' = \frac{z}{r} r'.$$

Therefore,

$$T = \frac{1}{2} m(x'^2 + y'^2 + z'^2) = \frac{1}{2} m(r'^2 + \omega^2 r^2 \sin^2 \alpha).$$

Also,

$$U = -mgz = -mgr \cos \alpha.$$

Hence, the Lagrangean function (Sec. 348) is

$$L = T + U = \frac{1}{2} m(r'^2 + \omega^2 r^2 \sin^2 \alpha) - mgr \cos \alpha.$$

The equation of motion (Eq. (348.2)) becomes

$$\left(\frac{\partial L}{\partial r'}\right)' - \frac{\partial L}{\partial r} = 0;$$

or, after removing the factor m, and developing,

$$r'' - r\omega^2 \sin^2 \alpha = -g \cos \alpha. \tag{2}$$

This is the equation of the motion of a particle along a fixed straight line subject to two forces; the first is one of repulsion from the origin; the second is a constant force always directed toward the negative end of the r-axis. The solution is

$$r = A \cosh (\omega t \sin \alpha) + B \sinh (\omega t \sin \alpha) + \frac{g \cos \alpha}{\omega^2 \sin^2 \alpha}.$$

If the initial conditions are

$$r = r_0, \qquad r' = 0,$$

then

$$A = r_0 - \frac{g \cos \alpha}{\omega^2 \sin^2 \alpha}, \qquad B = 0;$$

and the solution becomes

$$r = \left(r_0 - \frac{g \cos \alpha}{\omega^2 \sin^2 \alpha}\right) \cosh (\omega t \sin \alpha) + \frac{g \cos \alpha}{\omega^2 \sin^2 \alpha}.$$

For $\omega = 0$, or $\alpha = 0$, this solution reduces, as it should, to

$$r = r_0 - \frac{1}{2} g t^2 \cos \alpha.$$

The particle will rise or fall according as

$$r_0 \gtrless \frac{g \cos \alpha}{\omega^2 \sin^2 \alpha}.$$

Equation (2) admits the integral

$$r'^2 - r^2\omega^2 \sin^2 \alpha = -2gr \cos \alpha + C,$$

which is an illustration of the general integral (Eq. (349.6)).

356. The Particle is Constrained to a Moving Plane Surface.— A particle, which is acted upon by an attractive force which is proportional to its distance from a fixed point O, is constrained to move on a smooth plane which is in motion. The plane pivots on the point O, and its normal at O describes a cone with uniform angular motion. It is desired to find the motion of the particle.

Let ξ and η be a set of rectangular axes in the moving plane, with origin at O; the ξ-axis coinciding with the intersection of the

moving plane and the xy-plane. If α is the angle between the two planes, the equations of transformation are

$$\begin{aligned} x &= \xi \cos \omega t - \eta \cos \alpha \sin \omega t, \\ y &= \xi \sin \omega t + \eta \cos \alpha \cos \omega t, \\ z &= \qquad\qquad + \eta \sin \alpha; \end{aligned}$$

and their derivatives with respect to the time are

$$\begin{aligned} x' &= \xi' \cos \omega t - \eta' \cos \alpha \sin \omega t - \omega y, \\ y' &= \xi' \sin \omega t + \eta' \cos \alpha \cos \omega t + \omega x, \\ z' &= \qquad\qquad + \eta' \sin \alpha. \end{aligned}$$

Therefore,

$$x'^2 + y'^2 + z'^2 = \xi'^2 + \eta'^2 + 2\omega \cos \alpha(\xi\eta' - \eta\xi') + \omega^2(\xi^2 + \eta^2 \cos^2 \alpha).$$

The force function U can be written

$$U = -\frac{1}{2}mk^2(x^2 + y^2 + z^2) = -\frac{1}{2}mk^2(\xi^2 + \eta^2).$$

Hence, the Lagrangean function $T + U$ is

$$L = \frac{1}{2}m\{\xi'^2 + \eta'^2 + 2\omega \cos \alpha(\xi\eta' - \eta\xi') + \omega^2(\xi^2 + \eta^2 \cos^2 \alpha) - k^2(\xi^2 + \eta^2)\}.$$

From this it is easily found that

$$\frac{1}{m}\left(\frac{\partial L}{\partial \xi'}\right)' = \xi'' - \eta'\omega \cos \alpha, \qquad \frac{1}{m}\left(\frac{\partial L}{\partial \eta'}\right)' = \eta'' + \xi'\omega \cos \alpha,$$

and

$$\begin{aligned} -\frac{1}{m}\frac{\partial L}{\partial \xi} &= -\eta'\omega \cos \alpha + (k^2 - \omega^2)\xi, \\ -\frac{1}{m}\frac{\partial L}{\partial \eta} &= +\xi'\omega \cos \alpha + (k^2 - \omega^2 \cos^2 \alpha)\eta. \end{aligned}$$

The equations of motion are, therefore,

$$\begin{aligned} \xi'' - 2\omega \cos \alpha\eta' + (k^2 - \omega^2)\xi &= 0, \\ \eta'' + 2\omega \cos \alpha\xi' + (k^2 - \omega^2 \cos^2 \alpha)\eta &= 0. \end{aligned}$$

These equations are linear and homogeneous with constant coefficients. The solution is reduced, by means of the substitution

$$\xi = Ae^{\lambda t}, \qquad \eta = Be^{\lambda t}, \tag{1}$$

to the solution of two algebraic equations

$$\begin{aligned} A[\lambda^2 + (k^2 - \omega^2)] - B[2\lambda\omega \cos \alpha] &= 0, \\ A[2\lambda\omega \cos \alpha] + B[\lambda^2 + (k^2 - \omega^2 \cos^2 \alpha)] &= 0, \end{aligned} \tag{2}$$

which are linear and homogeneous in the constants A and B. In order that there may exist a solution, other than the trivial solution in which both A and B are zero, it is necessary that the determinant shall vanish, that is,

$$\begin{vmatrix} \lambda^2 + (k^2 - \omega^2), & -2\lambda\omega \cos \alpha \\ 2\lambda\omega \cos \alpha, & \lambda^2 + (k^2 - \omega^2 \cos^2 \alpha) \end{vmatrix} = 0,$$

or

$$\lambda^4 + [2k^2 + (3 \cos^2 \alpha - 1)\omega^2]\lambda^2 + (k^2 - \omega^2)(k^2 - \omega^2 \cos^2 \alpha) = 0. \quad (3)$$

The solution of this equation is

$$\lambda^2 = -k^2 + \frac{1}{2}\omega^2(1 - 3 \cos^2 \alpha) \quad (4)$$
$$\pm \frac{1}{2}\omega^2\sqrt{1 + \left(16\frac{k^2}{\omega^2} - 10\right) \cos^2 \alpha + 9 \cos^4 \alpha}.$$

Thus it is evident that the four values of λ occur in pairs, the two members of a pair differing only in sign. There are only two values of λ^2; and these two reduce to one if the radicand vanishes. For this exceptional case there are only two solutions of the form of Eq. (1). In the general case, however, Eq. (4) gives two distinct values of λ^2. As the nature of the motion depends upon whether the values of λ^2 are positive, negative, or complex, it is desirable to examine further the roots of Eq. (3). The two roots have the same or opposite signs according as

$$\left(\frac{k^2}{\omega^2} - 1\right)\left(\frac{k^2}{\omega^2} - \cos^2 \alpha\right) \gtrless 0.$$

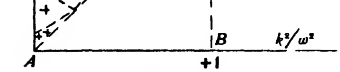


FIG. 176.

It is readily verified that if k^2/ω^2 is large both roots are negative (see Fig. 176). On crossing the line $k^2/\omega^2 = 1$, one of the roots changes sign, so that in the triangle ABD one root is positive and one is negative. In the triangle ACD the situation is more complicated, for the discriminant vanishes along the curve

$$1 + \left(16\frac{k^2}{\omega^2} - 10\right) \cos^2 \alpha + 9 \cos^4 \alpha = 0.$$

If

$$\frac{k^2}{\omega^2} = x > 0, \qquad \cos^2 \alpha = y \leqq +1,$$

this curve,

$$1 - 10y + 16xy + 9y^2 = 0,$$

which is a hyperbola through the point C, is asymptotic to the x-axis and to a line whose slope is $-16/9$, and to which the line $x = y$ is tangent. Thus the triangle ACD is divided into three regions. In the neighborhood of A both roots are positive. In the part cut out by the hyperbola, both roots are complex; and in the remainder of the triangle both roots are negative. In the region of complex roots, the real part of the root is positive if the point x, y is below the line

$$2x + 3y - 1 = 0,$$

and negative if the point lies above it. This line passes through the point of tangency of the line $x = y$ and the hyperbola.

If the four values of λ are denoted by λ_1, λ_2, λ_3, and λ_4, the complete solution is

$$\begin{aligned} \xi &= A_1 e^{\lambda_1 t} + A_2 e^{\lambda_2 t} + A_3 e^{\lambda_3 t} + A_4 e^{\lambda_4 t}, \\ \eta &= B_1 e^{\lambda_1 t} + B_2 e^{\lambda_2 t} + B_3 e^{\lambda_3 t} + B_4 e^{\lambda_4 t}; \end{aligned}$$

where, from Eq. (2),

$$B_i = \frac{\lambda_i^2 + (k^2 - \omega^2)}{2\lambda_i \cos \alpha} A_i.$$

If both values of λ^2 are negative, all four of the roots $\lambda_1, \cdots, \lambda_4$ are pure imaginaries, and the motion is compounded out of two simple harmonic motions. If one value of λ^2 is positive and one is negative, two of the roots are pure imaginaries and two are real, of which one is positive and one negative (equal numerically, however). The motion is compounded out of a simple harmonic motion and an exponential motion. The particle therefore, in general, recedes from the origin, but oscillates in doing so. If both roots are positive, all four exponentials are real and the particle, in general, recedes from the origin without oscillation. This is the case when the plane has a high inclination and revolves rapidly, and the attractive force is relatively feeble.

The integral Eq. (349.6) exists, since the Lagrangean function does not contain the time explicitly, namely,

$$\xi'^2 + \eta'^2 - \omega^2(\xi^2 + \eta^2 \cos^2 \alpha) + k^2(\xi^2 + \eta^2) = C.$$

The energy is not constant since the revolving plane does work upon the particle. While the reaction of the plane is always normal to the plane, it is not normal to the curve described by the particle in space, and therefore it does work upon the particle.

III. RELATIVE MOTION

357. Motion Relative to the Surface of the Earth.—When it is said that a system of rectangular axes is "fixed," it is meant that it is fixed relative to the center of gravity of the system of stars, or the galaxy; or, that it is in uniform translation with respect to such a system, since the differential equations of motion are the same for the two cases.

On account of both the annual revolution of the earth about the sun, and the daily rotation of the earth upon its own axis, a system of axes fixed relatively to the surface of the earth is not a system which is fixed in the sense which has just been defined. If the earth did not rotate upon its axis, the revolution about the sun would introduce merely a translation of the axis, the origin of the axis describing an ellipse about the sun annually. Such a translation is not a uniform translation, since the origin does not describe a straight line with uniform speed. The period of the motion (one year), however, is so large that no perceptible effect is produced in motions which occur on the surface of the earth; and therefore the departure from uniformity, due to this cause, generally can be ignored.

It is otherwise with the rotation of the earth, which has a period of 23 hours, 56 minutes, 4.1 seconds, or 86,164.1 mean solar seconds; for, not only is the period much shorter, but the axes, which are fixed relatively to the surface of the earth, are in rotation with respect to a system of fixed axes. It will be of interest, therefore, to see in what manner the apparent motions on the surface of the earth depend upon the rotation of the earth.

358. The Equations of Transformation.—Let a system of axes with the origin at the center of the earth and the z-axis coinciding with the axis of rotation be regarded as a system of fixed axes, the x- and y-axes in the plane of the equator having fixed directions with respect to the stars. With respect to these axes, the earth is rotating with the angular speed

$$\omega = \frac{2\pi}{86164.1} = 0.000072921,$$

which is so small that its square can generally be neglected. At the point P on the surface of the earth (Fig. 177) take a system of rectangular axes with the ζ-axis coinciding with the normal to the surface, the ξ-axis directed toward the east, and the η-axis directed toward the north. Let $\alpha_1, \cdots, \gamma_3$, as indicated

in the table, be the cosines of the angles which the axes of one system make with the axes of the other.

	ξ	η	ζ
x	α_1	α_2	α_3
y	β_1	β_2	β_3
z	γ_1	γ_2	γ_3

Let the polar coordinates of P with respect to the x-, y-, and z-axes be

$$x_0 = r \cos \varphi \cos \theta, \qquad y_0 = r \cos \varphi \sin \theta, \qquad z_0 = r \sin \varphi.$$

In accordance with the conventions of Sec. 350, the ξ-axis coincides with the θ-direction, the η-axis coincides with the φ-direction

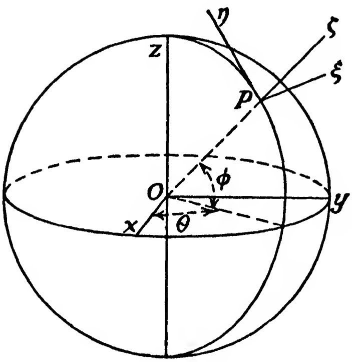


FIG. 177.

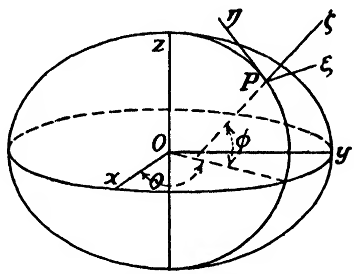


FIG. 178.

and the ζ-axis coincides with the r-direction. Hence, the expressions in Eq. (350.1) give

$$\begin{aligned}
\alpha_1 &= -\sin\theta, & \alpha_2 &= -\sin\varphi\cos\theta, & \alpha_3 &= \cos\varphi\cos\theta;\\
\beta_1 &= +\cos\theta, & \beta_2 &= -\sin\varphi\sin\theta, & \beta_3 &= \cos\varphi\sin\theta;\\
\gamma_1 &= 0, & \gamma_2 &= +\cos\varphi, & \gamma_3 &= \sin\varphi.
\end{aligned}$$

Since P is fixed on the surface of the earth, the angle φ is constant and θ is equal to ωt. Hence, the equations of transformation from one system to the other

$$\begin{aligned}
x &= x_0 + \alpha_1\xi + \alpha_2\eta + \alpha_3\zeta,\\
y &= y_0 + \beta_1\xi + \beta_2\eta + \beta_3\zeta,\\
z &= z_0 + \gamma_1\xi + \gamma_2\eta + \gamma_3\zeta
\end{aligned}$$

become

$$\begin{aligned}
x &= a\cos\varphi\cos\omega t - \xi\sin\omega t - \eta\sin\varphi\cos\omega t + \zeta\cos\varphi\cos\omega t,\\
y &= a\cos\varphi\sin\omega t + \xi\cos\omega t - \eta\sin\varphi\sin\omega t + \zeta\cos\alpha\sin\omega t,\\
z &= a\sin\varphi \quad + 0 \quad + \eta\cos\varphi \quad + \zeta\sin\varphi,
\end{aligned}$$

the radius of sphere having been taken equal to a.

The equations of transformation having been derived for the sphere, it is evident that the same equations hold for any surface of revolution provided φ is taken as the complement of the angle which the normal to the surface at P makes with the z-axis, and a is the distance from P to the z-axis measured along the normal. In particular, they hold if the earth is regarded as an oblate spheroid (Fig. 178) instead of a sphere.

The Components of Acceleration along the ξ-, η-, and ζ-axes.— The derivatives with respect to the time x', y', z' are

$$\begin{aligned}
x' &= x_0' + (\alpha_1'\xi + \alpha_2'\eta + \alpha_3'\zeta) + (\alpha_1\xi' + \alpha_2\eta' + \alpha_3\zeta'),\\
y' &= y_0' + (\beta_1'\xi + \beta_2'\eta + \beta_3'\zeta) + (\beta_1\xi' + \beta_2\eta' + \beta_3\zeta'),\\
z' &= z_0' + (\gamma_1'\xi + \gamma_2'\eta + \gamma_3'\zeta) + (\gamma_1\xi' + \gamma_2\eta' + \gamma_3\zeta');
\end{aligned}$$

and these values, substituted in the expression for the kinetic energy $T = m\,(x'^2 + y'^2 + z'^2)/2$ give,

$$\begin{aligned}
T &= \frac{1}{2}m(\xi'^2 + \eta'^2 + \zeta'^2)\\
&+ m\omega(\xi\eta' \sin\varphi - \xi\zeta' \cos\varphi - \eta\zeta' \sin\varphi + \zeta\xi' \cos\varphi + a\xi' \cos\varphi)\\
&+ m\omega^2\Big(\frac{1}{2}\xi^2 + \frac{1}{2}\eta^2 \sin^2\varphi + \frac{1}{2}\zeta^2 \cos^2\varphi - \eta\zeta \sin\varphi \cos\varphi\\
&\qquad - a\eta \sin\varphi \cos\varphi + a\zeta \cos^2\varphi + \frac{1}{2}a^2 \cos^2\varphi\Big).
\end{aligned}$$

The accelerations α_ξ, α_η, and α_ζ along the ξ-, η-, and ζ-axes are (Sec. 350)

$$\begin{aligned}
\alpha_\xi &= \frac{1}{m}\left[\left(\frac{\partial T}{\partial \xi'}\right)' - \frac{\partial T}{\partial \xi}\right],\\
\alpha_\eta &= \frac{1}{m}\left[\left(\frac{\partial T}{\partial \eta'}\right)' - \frac{\partial T}{\partial \eta}\right],\\
\alpha_\zeta &= \frac{1}{m}\left[\left(\frac{\partial T}{\partial \zeta'}\right)' - \frac{\partial T}{\partial \zeta}\right];
\end{aligned}$$

and the explicit expressions for these accelerations are

$$\left.\begin{aligned}
\alpha_\xi &= \xi'' + 2\omega(\xi' \cos\varphi - \eta' \sin\varphi) - \omega^2\xi,\\
\alpha_\eta &= \eta'' + 2\omega\xi' \sin\varphi + \omega^2 \sin\varphi(-\eta \sin\varphi + \zeta \cos\varphi +\\
&\qquad\qquad a \cos\varphi),\\
\alpha_\zeta &= \zeta'' - 2\omega\xi' \cos\varphi - \omega^2 \cos\varphi(-\eta \sin\varphi + \zeta \cos\varphi +\\
&\qquad\qquad a \cos\varphi).
\end{aligned}\right\} \quad (1)$$

359. A Vector Resolution of the Acceleration.—The terms which carry ω^2 as a factor in Eq. (358.1) are readily seen to be the negative of the components of the centrifugal acceleration of a

point ξ, η, ζ which is at rest relative to the ξ-, η-, and ζ-axes. In Fig. 179, let ζPOZ be the $\eta\zeta$-plane, which also passes through the z-axis.

Let the point C be the projection of the point ξ, η, ζ upon this plane. Let AC be the perpendicular from C to the z-axis, and PQ the perpendicular from P to the z-axis. Then

$$OP = a, \qquad PQ = a \cos \varphi, \qquad PE = \zeta, \qquad EC = \eta,$$

and

$$AC = a \cos \varphi + \zeta \cos \varphi - \eta \sin \varphi.$$

The components of the vector $\overrightarrow{AC}$ along the η- and ζ-axes are, respectively,

$$-(a \cos \varphi + \zeta \cos \varphi - \eta \sin \varphi) \sin \varphi$$

and

$$+(a \cos \varphi + \zeta \cos \varphi - \eta \sin \varphi) \cos \varphi.$$

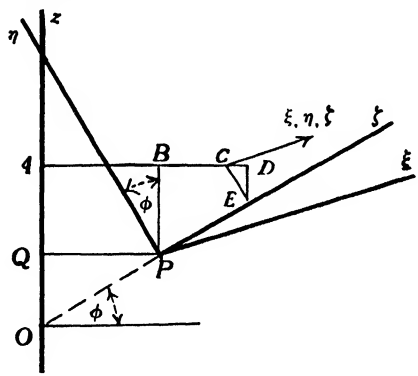


FIG. 179.

Let **L** be a vector perpendicular to the z-axis, with its origin in the z-axis and its terminus at the point ξ, η, ζ. Then $\omega^2\mathbf{L}$ is the centrifugal acceleration due to the rotation of the earth, and its components along the ξ-, η-, and ζ-axes are

$$\omega^2\xi, \qquad -\omega^2 \sin \varphi(-\eta \sin \varphi + \zeta \cos \varphi + a \cos \varphi),$$
$$+\omega^2 \cos \varphi(-\eta \sin \varphi + \zeta \cos \varphi + a \cos \varphi).$$

The terms which carry 2ω as a factor also are the components of a vector which is known as the *compound centrifugal acceleration*, since it depends not only upon rotation of the earth considered as a vector but also upon the relative velocity vector ξ', η', ζ'. In Fig. 180, the rotation of the earth is represented by the vector $\boldsymbol{\omega}$, and the relative velocity, of which the components are ξ', η', and ζ', by $\mathbf{v}$. The compound centrifugal acceleration is represented by the vector $\boldsymbol{\alpha}_{cc}$. Imagine the plane which contains the

vectors $\boldsymbol{\omega}$ and $\mathbf{v}$ to be rotated about the axis of $\boldsymbol{\omega}$ with the angular speed ω in the sense indicated by the vector $\boldsymbol{\omega}$, the angle $\widehat{\boldsymbol{\omega}\mathbf{v}}$ remaining constant. It will be shown that $\boldsymbol{\alpha}_{cc}$ is twice the velocity of the terminus of $\mathbf{v}$ in this rotation, and therefore is perpendicular to the plane which contains $\boldsymbol{\omega}$ and $\mathbf{v}$.

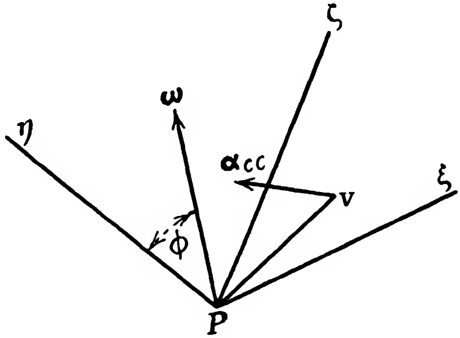


FIG. 180.

Let the direction cosines of $\boldsymbol{\omega}$ be α, β, and γ, the direction cosines of $\mathbf{v}$ be λ, μ, and ν, and the direction cosines of the velocity of the terminus of $\mathbf{v}$ in this rotation be a, b, and c. Then

$$\left.\begin{aligned} \alpha &= 0, & \beta &= \cos\varphi, & \gamma &= \sin\varphi, \\ \lambda &= \frac{\xi'}{v}, & \mu &= \frac{\eta'}{v}, & \nu &= \frac{\xi'}{v}. \end{aligned}\right\} \tag{1}$$

Since the velocity of the terminus of $\mathbf{v}$ is perpendicular both to $\mathbf{v}$ and $\boldsymbol{\omega}$, the following two equations hold:

$$a\alpha + b\beta + c\gamma = 0,$$
$$a\lambda + b\mu + c\nu = 0.$$

Therefore,

$$\frac{\nu\beta - \mu\gamma}{a} = \frac{\lambda\gamma - \nu\alpha}{b} = \frac{\mu\alpha - \lambda\beta}{c} \tag{2}$$
$$= \sqrt{(\nu\beta - \mu\gamma)^2 + (\lambda\gamma - \nu\alpha)^2 + (\mu\alpha - \lambda\beta)^2},$$

the last equality holding, since

$$a^2 + b^2 + c^2 = 1.$$

Now

$$\alpha\lambda + \beta\mu + \gamma\nu = \cos\widehat{\boldsymbol{\omega}\mathbf{v}},$$

and, therefore,

$$\sqrt{(\nu\beta - \mu\gamma)^2 + (\lambda\gamma - \nu\alpha)^2 + (\mu\alpha - \lambda\beta)^2} = \sin\widehat{\boldsymbol{\omega}\mathbf{v}}. \tag{3}$$

On substituting the values from Eqs. (1) and (3) in Eq. (2), and then multiplying through by ω, the components of the velocity of the terminus of $\mathbf{v}$ are found to be

$$\begin{aligned}
\omega v \sin \widehat{\boldsymbol{\omega}\mathbf{v}} \cdot a &= \omega(\xi' \cos \varphi - \eta' \sin \alpha),\\
\omega v \sin \widehat{\boldsymbol{\omega}\mathbf{v}} \cdot b &= \omega\xi' \sin \varphi,\\
\omega v \sin \widehat{\boldsymbol{\omega}\mathbf{v}} \cdot c &= -\omega\xi' \cos \varphi,
\end{aligned}$$

which are just one-half of the components of the compound centrifugal acceleration.

The terms ξ'', η'', and ζ'' are obviously the components of the *relative acceleration*, and therefore the total acceleration is resolvable into the three components: the relative acceleration, the centrifugal acceleration reversed, and the compound centrifugal acceleration. This proposition, which is known as the *theorem of Coriolis*, is true even though the ξ-, η-, and ζ-axes are not in simple rotation about a fixed axis. The present example is merely a particular instance of it.

360. The Plumb Line.—Let a sphere of mass m be suspended from a point on the ζ-axis by a string in which the tension is T. It will be supposed that the sphere is at rest at the origin. The plumb line will coincide with the ζ-axis, since by definition, the ζ-axis is vertical, and vertical means the direction of the plumb line. Let $\mathbf{G}$ be the acceleration of the gravitational attraction on the sphere. The earth is assumed to be symmetrical with respect to the $\eta\zeta$-plane, which is a meridian plane of the earth and, therefore, $\mathbf{G}$ will lie in this plane. Let its components along the η- and ζ-axes be $\mathbf{G}_\eta$ and $\mathbf{G}_\zeta$. Then the acceleration equations are

$$\left.\begin{aligned}
&\xi'' + 2\omega(\zeta' \cos \varphi - \eta' \sin \varphi) - \omega^2\xi = 0,\\
&\eta'' + 2\omega\xi' \sin \varphi\\
&\qquad\qquad + \omega^2 \sin \varphi(-\eta \sin \varphi + \zeta \cos \varphi + a \cos \varphi) = G_\eta,\\
&\zeta'' - 2\omega\xi' \cos \varphi\\
&\qquad - \omega^2 \cos \varphi(- \eta \sin \varphi + \zeta \cos \varphi + a \cos \varphi) = -G_\zeta + \frac{1}{m}T.
\end{aligned}\right\} \quad (1)$$

Since the sphere is relatively at rest at the origin,

$$\xi'' = \eta'' = \zeta'' = \xi' = \eta' = \zeta' = \xi = \eta = \zeta = 0;$$

and therefore

$$G_\eta = a\omega^2 \sin \varphi \cos \varphi, \qquad \frac{1}{m}T = G_\zeta - a\omega^2 \cos^2 \varphi.$$

If the earth were not rotating, the value of T/m would be G, and since

$$G_\zeta = \sqrt{G^2 - G_\eta^2} = G\sqrt{1 - \frac{a^2\omega^4}{G^2}\sin^2\varphi\cos^2\varphi}$$
$$= G - \frac{1}{2}\frac{a^2\omega^4}{G}\sin^2\varphi\cos^2\varphi + \cdots,$$

it is seen that T is slightly diminished by the rotation of the earth. The quantity $G_\zeta - a\omega^2\cos^2\varphi$ is the acceleration of gravity which is commonly denoted by the letter g.

The value of ω (Sec. 358) is approximately 1/14,000. This value is so small that, for regions in which g can be regarded as constant, the remaining terms in the acceleration equations which carry ω^2 as a factor are wholly inappreciable. They will therefore be dropped from further consideration.

361. Freely Falling Bodies.—The equations of motion for a freely falling body or a projectile, for a limited region in which it is permissible to regard g as a constant, are

$$\left.\begin{aligned} \xi'' + 2\omega(\zeta'\cos\varphi - \eta'\sin\varphi) &= 0, \\ \eta'' + 2\omega\xi'\sin\varphi &= 0, \\ \zeta'' - 2\omega\xi'\cos\varphi &= -g. \end{aligned}\right\} \quad (1)$$

Each of these equations is an exact differential. On integrating, there results

$$\left.\begin{aligned} \xi' + 2\omega(\zeta\cos\varphi - \eta\sin\varphi) &= c_1, \\ \eta' + 2\omega\xi\sin\varphi &= c_2, \\ \zeta' - 2\omega\xi\cos\varphi &= c_3 - gt. \end{aligned}\right\} \quad (2)$$

For a body falling freely from rest from a point on the ζ-axis at a height h above the origin,

$$c_1 = 2\omega h\cos\varphi, \qquad c_2 = 0, \qquad c_3 = 0.$$

From the second and third equations of Eq. (2), it is found that

$$(\zeta'\cos\varphi - \eta'\sin\varphi) = 2\omega\xi - g\cos\varphi\cdot t,$$

which, substituted in the first of Eq. (1), gives

$$\xi'' + 4\omega^2\xi = 2\omega g\cos\varphi\cdot t.$$

This last equation is easily integrated, and its solution is

$$\xi = c_4\cos 2\omega t + c_5\sin 2\omega t + \frac{g\cos\varphi}{2\omega}\cdot t.$$

By virtue of the initial conditions,

$$c_4 = 0, \qquad c_5 = -\frac{g\cos\varphi}{4\omega^2};$$

and therefore

$$\xi = \frac{g \cos \varphi}{2\omega}\left(t - \frac{\sin 2\omega t}{2\omega}\right)$$
$$= \frac{1}{3}\omega g \cos \varphi \cdot t^3 + \omega^3(\ \cdot\ \cdot\ \cdot\ ).$$

The integration of the second and third equations of Eq. (2) is now easily effected. If the terms which carry ω^2 as a factor are neglected, the solution which satisfies the initial conditions is

$$\xi = \frac{1}{3}\omega g \cos \varphi \cdot t^3, \qquad \eta = 0, \qquad \zeta = h - \frac{1}{2}gt^2.$$

The trajectory, therefore, lies in the $\xi\zeta$-plane, or the *prime vertical*, as it is called by the astronomers. Its equation is

$$\xi = \frac{4}{3}\frac{\omega \cos \varphi}{\sqrt{2g}}(h - \zeta)^{\frac{3}{2}},$$

which is the equation of a semicubical parabola. The body falls toward the east, and in latitude 40° the amount of the deviation is $0.00000928 \times h^{\frac{3}{2}}$ feet, if h is expressed in feet. For a drop of 1000 feet, the eastward deviation is 3.52 inches.

It is useless to compute the terms in ω^2. In order to give them any significance, it would be necessary to take into account not only the changes in the value of g due to changes in height, but also the attraction of the moon and irregularities in the density of the earth.

362. The Foucault Pendulum.—The Foucault pendulum differs from an ordinary spherical pendulum in that the Foucault pendulum is started from a position of rest relative to the surface of the earth, and the effects of the rotation of the earth are taken into account. Let the pendulum be of length l and be suspended from the origin of the ξ-, η-, and ζ-axes. Let the tension in the suspending wire be denoted by $m\mathbf{T}$, m being the mass of the pendulum bob. The components of $\mathbf{T}$ along the coordinate axes are $-T\xi/l$, $-T\eta/l$, and $-T\zeta/l$; and, therefore, the equations of motion are

$$\left.\begin{aligned} \xi'' + 2\omega(\zeta' \cos \varphi - \eta' \sin \varphi) &= -T\frac{\xi}{l}, \\ \eta'' + 2\omega\xi' \sin \varphi &= -T\frac{\eta}{l}, \\ \zeta'' - 2\omega\xi' \cos \varphi &= -T\frac{\zeta}{l} - g. \end{aligned}\right\} \tag{1}$$

The equation of constraint, of course, is

$$\xi^2 + \eta^2 + \zeta^2 = l^2.$$

If ω were zero, these would be the equations of motion for the spherical pendulum. Their solution is therefore more difficult and the motion is more complicated, if one considers the general case. Indeed, it is not known how to integrate them in finite terms for the general case. But if the motion is restricted to very small (infinitesimal) oscillations about the position of equilibrium, the equations of motion can be very much simplified.

Let ξ/l, η/l, and their derivatives be regarded as small quantities of the first order, so that their squares and cross-products and terms of higher order can be neglected. Then

$$\zeta = -l\left[1 - \frac{\xi^2 + \eta^2}{l^2}\right]^{\frac{1}{2}}$$

reduces to

$$\zeta = -l,$$

which is constant. The third equation then gives

$$T = g - 2\omega\xi' \cos \varphi;$$

or, effectively,

$$T = g,$$

since, when this value of T is introduced into the first two equations, the second term of T merely introduces a second-order term which would be dropped. On writing

$$\omega_1 = \omega \sin \varphi \tag{2}$$

for simplicity of notation, and dropping second-order terms, the first two equations become

$$\left.\begin{aligned} \xi'' - 2\omega_1\eta' &= -\frac{g}{l}\xi, \\ \eta'' + 2\omega_1\xi' &= -\frac{g}{l}\eta. \end{aligned}\right\} \tag{3}$$

Let the motion be referred to axes which are rotating backward with the constant angular speed ω_1, by means of the transformation

$$\left.\begin{aligned} \xi &= +\xi_1 \cos \omega_1 t + \eta_1 \sin \omega_1 t, \\ \eta &= -\xi_1 \sin \omega_1 t + \eta_1 \cos \omega_1 t. \end{aligned}\right\} \tag{4}$$

Then Eqs. (3) become

$$\xi_1'' + \left(\omega_1^2 + \frac{g}{l}\right)\xi_1 = 0,$$

$$\eta_1'' + \left(\omega_1^2 + \frac{g}{l}\right)\eta_1 = 0,$$

or, more simply,

$$\xi_1'' + \frac{g}{l}\xi_1 = 0, \qquad \eta'' + \frac{g}{l}\eta_1 = 0. \tag{5}$$

Thus, whatever the initial conditions may be, the motion with respect to the rotating axes is simple harmonic with the period

$$P_1 = 2\pi\sqrt{\frac{l}{g}},$$

just as for the simple pendulum. The period of the rotating axes

$$P = \frac{2\pi}{\omega_1} = \frac{2\pi}{\omega \sin \varphi} = \frac{24^h}{\sin \varphi} \text{ (sidereal time)} \tag{6}$$

depends only upon the latitude and not at all upon the initial conditions or the length and weight of the pendulum.

In the Foucault experiment the pendulum is drawn out of the vertical and released from rest relative to the surface of the earth. That is, initially, $\xi' = \eta' = 0$; and from Eq. (4) it is learned that this compels

$$\xi_1'(0) = -\omega_1\eta_1(0), \qquad \eta_1' = +\omega_1\xi_1(0). \tag{7}$$

The areas integral of Eq. (5) is

$$\begin{aligned} \xi_1\eta_1' - \eta_1\xi_1' &= \text{constant} = \frac{2\pi ab}{P_1}, \\ &= \omega_1(\xi_1^2(0) + \eta_1^2(0)) \qquad \text{by Eq. (7)}, \\ &= \omega_1 a^2; \end{aligned}$$

where a is the major axis and b is the minor axis of the ellipse. Hence,

$$a^2\omega_1 = \frac{2\pi ab}{P_1};$$

and on replacing the value of ω_1 from Eq. (6), this becomes

$$\frac{a}{P} = \frac{b}{P_1}.$$

This is the theorem of Chevilliet. The major and minor axes of the ellipse are proportional, respectively, to the period of the rotating axes and the natural period of the pendulum.

In order to keep the pendulum in motion for as long a time as possible, it is desirable to have the bob as heavy and the wire as long as possible. In the original experiment of Foucault in the Pantheon at Paris in 1851,

$$l = 67 \text{ meters}, \ a = 3^m, \ \frac{b}{a} = \frac{1}{7200}, \ P_1 = 16^s, \text{ and } P = 32^h.$$

Problems XXIII

1. By means of Eq. (361.2) show that a projectile which has a flat trajectory seems to deviate toward the right in the northern hemisphere and toward the left in the southern hemisphere. (Ferel's law.)

2. A heavy pendulum bob is suspended by a light rod of length l. The point of suspension of the rod describes a circle of radius a with the uniform angular speed ω. Show that if θ is zero when t is zero, the equation of motion of the pendulum is

$$\theta'' + \frac{a}{l}\omega^2 \sin(\theta - \omega t) + \frac{g}{l}\sin\theta = 0.$$

3. According to the method of Lagrange, the equations of motion of a particle which moves upon the helicoidal surface

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad z = b\theta,$$

and which is repelled from the axis by a force which is proportional to the distance from the axis, are

$$r'' - r\theta'^2 = k^2 r, \qquad ([b^2 + r^2]\theta')' = 0.$$

4. A particle of mass m is constrained to move on a smooth plane which turns with uniform motion about a horizontal axis which is taken as the x-axis. Aside from the constraint it is subject to no force except that of gravity. Show that the projection of the path of the particle upon the yz-plane is the curve

$$r = A\sinh(\theta - \theta_0) + \frac{g}{2\omega^2}\sin\theta.$$

5. If the origin is a position of stable equilibrium for a particle upon a surface under the action of gravity, so that the equation of the surface near the origin is

$$z = \frac{x^2}{2a} + \frac{y^2}{2b} + \text{higher degree terms,}$$

and if the oscillations of the particle about this position of equilibrium are very small (infinitesimal), the coordinates of the particle are given by the equations

$$x = A\cos\left(\sqrt{\frac{g}{a}}\cdot t + c_1\right), \qquad y = B\cos\left(\sqrt{\frac{g}{b}}\cdot t + c_2\right).$$

The projection of the path on the xy-plane is an algebraic curve if a and b are commensurable. If they are incommensurable, the path fills the entire rectangle of sides $2A$ and $2B$, in the sense that it passes any given point within the rectangle x_1, y_1 infinitely many times within a given distance $\epsilon > 0$, however small ϵ may be.

6. A circular hoop turns with uniform angular motion about an axis which passes through the circumference of the hoop, and is perpendicular to the plane of the hoop. A bead slides freely on the hoop, subject to no force save the constraint of the hoop. Show that the motion of the bead relative to the hoop is the same as the motion of a simple pendulum, and that for infinitesimal oscillations it has the same period as the hoop itself.

CHAPTER XV

THE CANONICAL EQUATIONS OF HAMILTON

363. Introduction.—The equations of Lagrange,

$$\left(\frac{\partial T}{\partial q_i'}\right)' - \frac{\partial T}{\partial q_i} = \frac{\partial U}{\partial q_i} \qquad i = 1, 2, 3, \tag{1}$$

in a sense, are symbolical only, since $\partial T/\partial q_i'$ is not itself one of the variables. Inasmuch as T is a quadratic function of q_1', q_2', q_3', these equations, when written out explicitly, are three differential equations, each of the second order, and each linear in q_1'', q_2'', q_3''.

It was Poisson who first suggested taking the partial derivatives of T with respect to the q_i' as new variables; but it was Hamilton who carried the transformation to its conclusion. The resulting equations are called *canonical* on account of their simplicity of form. They are not of particular advantage in the solution of elementary problems, such as those which have heretofore been considered. They have been employed to advantage, particularly by Poincaré, in long and difficult investigations in the domain of celestial mechanics and mathematical physics, where changes of variables are frequent. A knowledge of this form of the equations is, therefore, indispensable to anyone who would pursue these subjects.

364. The Equations of Transformation Do Not Contain the Time Explicitly.—The kinetic energy of a particle of mass m expressed in terms of the rectangular coordinates is

$$T = \frac{1}{2}m(x'^2 + y'^2 + z'^2). \tag{1}$$

If the equations of transformation (Eq. (347.2))

$$x = \varphi_1(q_1, q_2, q_3), \quad y = \varphi_2(q_1, q_2, q_3), \quad z = \varphi_3(q_1, q_2, q_3) \tag{2}$$

do not contain the time explicitly, the transformed expression for the kinetic energy is a homogeneous quadratic form in q_1', q_2', q_3'.

It is assumed, of course, that in the equations of transformation, the functions φ_1, φ_2, and φ_3, are independent functions. That is to

say, there does not exist a relation, which is independent of q_1, q_2, q_3, explicitly, of the form

$$\Phi(\varphi_1, \varphi_2, \varphi_3) \equiv 0,$$

in the arguments q_1, q_2, and q_3. If such a relation existed, the rectangular coordinates would not be independent, and the particle would not be free. If the notation

$$\frac{\partial \varphi_i}{\partial q_j} = \varphi_{ij} \qquad i, j = 1, 2, 3$$

be used, this condition is expressed by the non-vanishing of the functional determinant;[1] that is,

$$\Delta = \begin{vmatrix} \varphi_{11} & \varphi_{12} & \varphi_{13} \\ \varphi_{21} & \varphi_{22} & \varphi_{23} \\ \varphi_{31} & \varphi_{32} & \varphi_{33} \end{vmatrix} \neq 0. \tag{3}$$

It will be assumed hereafter that this condition is satisfied.

The derivatives of x, y, z are

$$\left.\begin{aligned} x' &= \varphi_{11}q_1' + \varphi_{12}q_2' + \varphi_{13}q_3', \\ y' &= \varphi_{21}q_1' + \varphi_{22}q_2' + \varphi_{23}q_3', \\ \text{and} \qquad z' &= \varphi_{31}q_1' + \varphi_{32}q_2' + \varphi_{33}q_3'. \end{aligned}\right\} \tag{4}$$

Therefore, the expression for T in terms of the new variables is

$$\begin{aligned} T = \frac{1}{2}m[(\varphi^2{}_{11} + \varphi^2{}_{21} + \varphi^2{}_{31})q'^2{}_1 + 2(\varphi_{11}\varphi_{12} + \varphi_{21}\varphi_{22} \\ + \varphi_{31}\varphi_{32})q_1'q_2' + 2(\varphi_{11}\varphi_{13} + \varphi_{21}\varphi_{23} + \varphi_{31}\varphi_{33})q_1'q_3' + (\varphi^2{}_{12} \\ + \varphi^2{}_{22} + \varphi^2{}_{32})q'^2{}_2 + 2(\varphi_{12}\varphi_{13} + \varphi_{22}\varphi_{23} + \varphi_{32}\varphi_{33})q_2'q_3' \\ + (\varphi^2{}_{13} + \varphi^2{}_{23} + \varphi^2{}_{33})q'^2{}_3]. \end{aligned} \tag{5}$$

365. Introduction of New Dependent Variables.—Let new variables p_1, p_2, and p_3 be introduced by the relations

$$\frac{\partial T}{\partial q_1'} = p_1, \qquad \frac{\partial T}{\partial q_2'} = p_2, \qquad \frac{\partial T}{\partial q_3'} = p_3, \tag{1}$$

so that

$$\left.\begin{aligned} \frac{p_1}{m} &= [(\varphi^2{}_{11} + \varphi^2{}_{21} + \varphi^2{}_{31})q_1' + (\varphi_{11}\varphi_{12} + \varphi_{21}\varphi_{22} + \varphi_{31}\varphi_{32})q_2' \\ &\qquad + (\varphi_{11}\varphi_{13} + \varphi_{21}\varphi_{23} + \varphi_{31}\varphi_{33})q_3'], \\ \frac{p_2}{m} &= [(\varphi_{11}\varphi_{12} + \varphi_{21}\varphi_{22} + \varphi_{31}\varphi_{32})q_1' + (\varphi^2{}_{12} + \varphi^2{}_{22} + \varphi^2{}_{32})q_2' \\ &\qquad + (\varphi_{12}\varphi_{13} + \varphi_{22}\varphi_{23} + \varphi_{32}\varphi_{33})q_3'], \\ \frac{p_3}{m} &= [(\varphi_{11}\varphi_{13} + \varphi_{21}\varphi_{23} + \varphi_{31}\varphi_{33})q_1' + (\varphi_{12}\varphi_{13} + \varphi_{22}\varphi_{23} \\ &\qquad + \varphi_{32}\varphi_{33})q_2' + (\varphi^2{}_{13} + \varphi^2{}_{23} + \varphi^2{}_{33})q_3']. \end{aligned}\right\} \tag{2}$$

[1] GOURSAT-HEDRICK, "Mathematical Analysis," vol. I, p. 52.

These equations are linear not only in the letters p_1, p_2, and p_3, but also in the accented letters q_1', q_2', and q_3'. They can be solved, therefore, for q_1', q_2', q_3' in terms of p_1, p_2, p_3, provided the determinant of Eq. (2) is not zero. It is readily verified, however, that this determinant is the square of Δ (Eq. (364.3)) which by hypothesis is not zero. The solution of Eq. (2), therefore, is always possible, and

$$\left.\begin{aligned} mq_1' &= A_{11}p_1 + A_{12}p_2 + A_{13}p_3, \\ mq_2' &= A_{21}p_1 + A_{22}p_2 + A_{23}p_3, \\ mq_3' &= A_{31}p_1 + A_{32}p_2 + A_{33}p_3; \end{aligned}\right\} \tag{3}$$

where the letters A_{ij} are functions of q_1, q_2, q_3. These expressions for q_1', q_2', q_3', substituted in Eq. (364.5), make T a homogeneous quadratic form in p_1, p_2, p_3 with coefficients B_{ij} which are functions of q_1, q_2, q_3; namely,

$$\begin{aligned} T = \frac{1}{2m}[B_{11}p_1^2 &+ 2B_{12}p_1p_2 + 2B_{13}p_1p_3 \\ &+ B_{22}p_2^2 + 2B_{23}p_2p_3 + B_{33}p_3^2]. \end{aligned} \tag{4}$$

The significance of the quantities p_1, p_2, and p_3 can be obtained as follows: The components of the momentum of the particle in the x-, y-, and z-directions are

$$\begin{aligned} mx' &= m[\varphi_{11}q_1' + \varphi_{12}q_2' + \varphi_{13}q_3'], \\ my' &= m[\varphi_{21}q_1' + \varphi_{22}q_2' + \varphi_{23}q_3'], \\ mz' &= m[\varphi_{31}q_1' + \varphi_{32}q_2' + \varphi_{33}q_3']. \end{aligned}$$

The component of the momentum in the q_1-direction, which can be denoted by M_1, is obtained by multiplying these equations, respectively, by φ_{11}/R_1, φ_{21}/R_1, and φ_{31}/R_1 which, by Sec. 350, are the direction cosines of the q_1-direction, and then adding. The result is

$$\begin{aligned} M_1 = \frac{m}{R_1}[(\varphi^2{}_{11} + \varphi^2{}_{21} + \varphi^2{}_{31})q_1' &+ (\varphi_{11}\varphi_{12} + \varphi_{21}\varphi_{22} + \varphi_{31}\varphi_{32})q_2' \\ &+ (\varphi_{11}\varphi_{13} + \varphi_{21}\varphi_{23} + \varphi_{31}\varphi_{33})q_3']. \end{aligned}$$

Therefore,

$$p_1 = R_1M_1;$$

and similarly for p_2 and p_3. That is, the quantities p_i are the components of the momentum of the particle in the q_i-directions multiplied by the quantities R_i. For this reason the p_i are called the *generalized momenta*.

366. Transformation of the Equations of Lagrange to the Canonical Form of Hamilton.—The kinetic energy is now

expressed in two different ways. In Eq. (364.5) the variables are $q_1, q_2, q_3; q_1', q_2', q_3'$. Accordingly, when it is desired to call attention to the fact that these are the variables which are used, the kinetic energy will be denoted by T_5. In Eq. (365.4), however, the variables are $q_1, q_2, q_3; p_1, p_2, p_3$ and, when it is desired to call attention to the fact that these are the variables, the kinetic energy will be denoted by T_4. With this understanding, the equations of Lagrange are

$$\left(\frac{\partial T_5}{\partial q_i'}\right)' - \frac{\partial T_5}{\partial q_i} = Q_i \qquad i = 1, 2, 3$$

and, therefore, by Eq. (365.1),

$$p_i' - \frac{\partial T_5}{\partial q_i} = Q_i. \tag{1}$$

Regarding $q_1, q_2, q_3; q_1', q_2', q_3'$ as independent variables, the total differential of T is

$$dT = \sum_{i=1}^{3} \frac{\partial T_5}{\partial q_i} dq_i + \sum_{i=1}^{3} \frac{\partial T_5}{\partial q_i'} dq_i';$$

or

$$dT = \sum_{i=1}^{3} \frac{\partial T_5}{\partial q_i} dq_i + \sum_{i=1}^{3} p_i dq_i'. \tag{2}$$

Regarding $q_1, q_2, q_3; p_1, p_2, p_3$ as the independent variables,

$$dT = \sum_{i=1}^{3} \frac{\partial T_4}{\partial q_i} dq_i + \sum_{i=1}^{3} \frac{\partial T_4}{\partial p_i} dp_i. \tag{3}$$

Also, since T is a homogeneous function of the second degree,

$$2T = \sum_{i=1}^{3} q_i' \frac{\partial T_5}{\partial q_i'} = \sum_{i=1}^{3} p_i q_i', \tag{4}$$

and therefore

$$2dT = \sum_{i=1}^{3} p_i dq_i' + \sum_{i=1}^{3} q_i' dp_i. \tag{5}$$

On subtracting Eq. (2) from Eq. (5), there results

$$dT = -\sum_{i=1}^{3} \frac{\partial T_5}{\partial q_i} dq_i + \sum_{i=1}^{3} q_i' dp_i. \tag{6}$$

Bearing in mind that the differentials of $q_1, q_2, q_3; p_1, p_2, p_3$ are entirely independent, a comparison of Eqs. (3) and (6) shows that

$$\frac{\partial T_4}{\partial q_i} = -\frac{\partial T_5}{\partial q_i}, \qquad \frac{\partial T_4}{\partial p_i} = q_i'. \tag{7}$$

Then from Eqs. (1) and (7), the following are derived:

$$q_i' = \frac{\partial T_4}{\partial p_i}, \qquad p_i' = -\frac{\partial T_4}{\partial q_i} + Q_i, \qquad i = 1, 2, 3. \tag{8}$$

If there exists a force function $U(q_1, q_2, q_3; t)$, which may contain the time explicitly, but not q_1', q_2', q_3', such that

$$Q_i = \frac{\partial U}{\partial q_i},$$

then Eq. (8) becomes (dropping the subscript on T)

$$q_i' = \frac{\partial T}{\partial p_i}, \qquad p_i' = -\frac{\partial T}{\partial q_i} + \frac{\partial U}{\partial q_i}. \tag{9}$$

Now let

$$T - U = H.$$

The function H is called the Hamiltonian function, just as

$$T + U = L,$$

in Sec. 348, was called the Lagrangean function. The function U does not depend upon p_1, p_2, p_3, so that

$$\frac{\partial T}{\partial p_i} = \frac{\partial T}{\partial p_i} - \frac{\partial U}{\partial p_i}.$$

Therefore Eq. (9) becomes, on using the Hamiltonian function,

$$q_i' = \frac{\partial H}{\partial p_i}, \qquad p_i' = -\frac{\partial H}{\partial q_i}, \qquad i = 1, 2, 3. \tag{10}$$

These are the canonical equations which were sought. In the Lagrangean set, there are three equations each of the second order; while in the canonical set there are six equations each of the first order. In forming the Hamiltonian function, it must not be forgotten that the kinetic energy has the form of Eq. (365.4) and not that of Eq. (364.5).

367. Removal of the Restriction.—The preceding argument rests upon the assumption that the equations of transformation (Eq. (364.2)) do not contain the time explicitly. As a matter of fact, this assumption was made merely that the argument might be as simple and as clear as possible. It made all the equations homogeneous and, therefore, relatively simple. The removal of the restriction makes the equations non-homogeneous and, therefore, more complicated, but it does not alter the course of the argument.

In the first place, the functional determinant (Eq. (364.3)) of the equations of transformation (Eq. (364.2)) under the assump-

tion that they contain the time explicitly, remains unaltered. The equations in Eq. (364.4) become non-homogeneous by the addition of a term φ_{10}, φ_{20}, φ_{30} to the first, second, and third equations, respectively, where

$$\varphi_{i0} = \frac{\partial \varphi_i}{\partial t}.$$

The expression for the kinetic energy (Eq. (364.5)) becomes non-homogeneous, and can be written

$$T = T_2 + T_1 + T_0, \tag{1}$$

where T_2 is the sum of the terms which are homogeneous of degree 2, and therefore identical with the terms written explicitly in Eq. (364.5); T_1 are the terms which are homogeneous of degree 1 in q_1', q_2', q_3'; and T_0 is the sum of the terms which are independent of q_1', q_2', q_3'.

In Sec. 365, Eq. (1) remains unaltered. Equation (2) becomes non-homogeneous, but the terms already written remain unaltered. The condition for the possibility of the solution of Eq. (365.2) for the q_i' in terms of the p_i depends, not upon their homogeneity or non-homogeneity, but upon the determinant of the linear terms. This determinant is unaltered and has the same significance as before. Therefore, this solution is always possible. Equations (365.3) and (365.4) become non-homogeneous; and so also do the equations for momenta which follow, but the significance of p_1, p_2, p_3 remains unaltered, in the sense that

$$p_i = R_i M_i.$$

Equations (366.1), (366.2), and (366.3) are unaltered; but Eq. (366.4) becomes

$$\sum_{i=1}^{3} q_i' \frac{\partial T_5}{\partial q_i'} = \sum_{i=1}^{3} p_i q_i' = 2T_2 + T_1. \tag{2}$$

Therefore, taking $D_5 = T_2 - T_0$,

$$D_5 = \sum_{i=1}^{3} p_i q_i' - T_5,$$

the subscript 5 having the same significance as before. Differentiating,

$$dD_5 = \sum q_i' dp_i + \left(\sum p_i dq_i' - \sum \frac{\partial T_5}{\partial q_i'} dq_i'\right) - \sum \frac{\partial T_5}{\partial q_i} dq_i$$

which, since the terms within the parenthesis vanish, becomes

$$dD_5 = \sum q_i' dp_i - \sum \frac{\partial T_5}{\partial q_i} dq_i. \tag{3}$$

On replacing the letters q_i' in D_5 by their values in terms of p_1, p_2, p_3, D_5 becomes D_4 with the arguments $q_1, q_2, q_3; p_1, p_2, p_3$. On differentiating D_4, however,

$$dD_4 = \sum \frac{\partial D_4}{\partial p_i} dp_i + \sum \frac{\partial D_4}{\partial q_i} dq_i. \tag{4}$$

The differentials

$$dq_1,\ dq_2,\ dq_3;\ dp_1,\ dp_2,\ dp_3$$

are six arbitrary differentials, but whatever their values may be

$$dD_5 \equiv dD_4.$$

Therefore, on comparing Eqs. (3) and (4), it is seen that

$$q_i' = \frac{\partial D_4}{\partial p_i}, \qquad \frac{\partial T_5}{\partial q_i} = -\frac{\partial D_4}{\partial q_i}. \tag{5}$$

Then, from Eqs. (366.1) and (5),

$$q_i' = \frac{\partial D_4}{\partial p_i}, \qquad p_i' = -\frac{\partial D_4}{\partial q_i} + Q_i.$$

If a potential function $U(q_1, q_2, q_3; t)$ exists, and if

$$H = D_4 - U = T_2 - T_0 - U, \tag{6}$$

then, just as before,

$$\left.\begin{aligned} q_1' &= +\frac{\partial H}{\partial p_1}, & q_2' &= +\frac{\partial H}{\partial p_2}, & q_3' &= +\frac{\partial H}{\partial p_3}, \\ p_1' &= -\frac{\partial H}{\partial q_1}, & p_2' &= -\frac{\partial H}{\partial q_2}, & p_3' &= -\frac{\partial H}{\partial q_3}, \end{aligned}\right\} \tag{7}$$

which are Hamilton's equations for a free particle without any restrictions save that the equations of transformation are holonomic (Sec. 347).

368. The Energy Integral.—If the potential function U does not contain the time and if the equations of transformation (Eq. (364.2)) also are independent of the time, then the expressions for T and also H (Sec. 366) are free from the time. If the first equation of Eq. (366.10) is multiplied by p_i' and the second by $-q_i'$ and the two equations are added, and then summed with respect to the index i, there results

$$\sum_{i=1}^{3} \frac{\partial H}{\partial p_i} p_i' + \sum_{i=1}^{3} \frac{\partial H}{\partial q_i} q_i' = 0;$$

that is, since H does not contain t explicitly,

$$\frac{dH}{dt} = 0, \qquad \therefore H = \text{constant}.$$

But it is known already that, under these hypotheses,

$$H = T - U$$

is the expression for the energy. This merely reaffirms, in a new setting, the already familiar fact that the energy is constant.

It can happen, even under the hypotheses of Sec. 367, that H is free from the time explicitly. When this is true, the equations in Eq. (367.7) admit the same integral as the equations in Eq. (366.10), namely,

$$H = \text{constant};$$

but in this case $H = T_2 - T_0 - U$ is not the energy. It can, perhaps, be regarded as the energy relative to a moving set of axes, for which the potential energy is $-(T_0 + U)$ and T_2 is the kinetic energy.

369. Constrained Motion.—If the particle is not free but is constrained to move on a surface, and if the coordinates q_1 and q_2 are suitably chosen, the equations of transformation become

$$x = \varphi_1(q_1, q_2), \qquad y = \varphi_2(q_1, q_2), \qquad z = \varphi_3(q_1, q_2). \qquad (1)$$

The equation of the surface is obtained by eliminating q_1 and q_2 between these three equations. So far as mere form is concerned, the equations in Eq. (1) are the same as the equations in Eq. (364.2), in which $q_3 \equiv 0$. The functional determinant (Eq. (364.3)) vanishes identically since each element in the last column is zero. But not all of the minors formed from the first two columns are zero; for if they were there would exist not merely one relation between x, y, z, but two such relations, and the particle would be constrained to move on a line. Assuming that the particle is free upon a single surface then not all of the minors formed from the first two columns of Eq. (364.3) are zero.

Let the minor of φ_{i3} in Eq. (364.3) be Φ_{i3}. The equations in Eq. (365.2) are reduced to

$$\left.\begin{aligned} p_1 &= m[(\varphi^2_{11} + \varphi^2_{21} + \varphi^2_{31})q_1' \\ &\qquad\qquad + (\varphi_{11}\varphi_{12} + \varphi_{21}\varphi_{22} + \varphi_{31}\varphi_{32})q_2'] \\ p_2 &= m[(\varphi_{11}\varphi_{12} + \varphi_{21}\varphi_{22} + \varphi_{31}\varphi_{32})q_1' \\ &\qquad\qquad + (\varphi^2_{12} + \varphi^2_{22} + \varphi^2_{32})q_2'], \end{aligned}\right\} \qquad (2)$$

since there are now but two p's instead of three. The determinant of the right member is equal to

$$\Delta = \Phi^2_{13} + \Phi^2_{23} + \Phi^2_{33}$$

which cannot vanish, for by hypothesis, not all of the first minors of Eq. (364.3) are zero. Hence, Eq. (2) can be solved for q_1' and q_2' in terms of p_1 and p_2. The remainder of the argument proceeds just as in Sec. 367, for nowhere else does the argument depend upon the number of the coordinates. The equations of motion for a constrained particle, therefore, can be put in the canonical form, just as they can for a free particle.

370. An Equivalent Form of the Equations.—Suppose that the equations in Eq. (367.7) have been completely integrated. Then $q_1, q_2, q_3; p_1, p_2, p_3$ will be expressed as functions of t; $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6$, where the α's are the six constants of integration. The functional determinant of the p's and q's with respect to the α's is not zero, for if it were there would exist a relation between $q_1, q_2, q_3; p_1, p_2, p_3; t$, independent of the α's, and it would not be possible to choose the initial values arbitrarily. If the values of the p's and q's in terms of t and the α's be substituted in H, then H, too, becomes a function of t and the α's. Differentiating H with respect to any one of these constants, there results

$$\frac{\partial H}{\partial \alpha_k} = \sum_{i=1}^{3} \frac{\partial H}{\partial p_i}\frac{\partial p_i}{\partial \alpha_k} + \sum_{i=1}^{3} \frac{\partial H}{\partial q_i}\frac{\partial q_i}{\partial \alpha_k} \qquad k = 1, \cdots, 6. \qquad (1)$$

There exists also the identity

$$\frac{d}{dt}\sum_{i=1}^{3} q_i \frac{\partial p_i}{\partial \alpha_k} - \frac{\partial}{\partial \alpha_k}\sum_{i=1}^{3} q_i p_i' \equiv \sum_{i=1}^{3} \frac{\partial p_i}{\partial \alpha_k} q_i' - \sum_{i=1}^{3} \frac{\partial q_i}{\partial \alpha_k} p_i'. \qquad (2)$$

If the values of p_i' and q_i' in Eq. (367.7) be substituted in the right member of Eq. (2), then, in view of Eq. (1), Eq. (2) becomes

$$\frac{d}{dt}\sum_{i=1}^{3} q_i \frac{\partial p_i}{\partial \alpha_k} - \frac{\partial}{\partial \alpha_k}\sum_{i=1}^{3} q_i p_i' = \frac{\partial H}{\partial \alpha_k} \qquad k = 1, \cdots, 6. \qquad (3)$$

Thus, if the equations in Eq. (367.7) are true, the equations in Eq. (3) are true.

Conversely, if the equations in Eq. (3) are true, then the equations in Eq. (367.7) are true. For, the left members of Eq. (2) and (3) being identical, the right members are equal, that is,

$$\frac{\partial H}{\partial \alpha_k} = \sum_{i=1}^{3} \frac{\partial p_i}{\partial \alpha_k} q_i' - \sum_{i=1}^{3} \frac{\partial q_i}{\partial \alpha_k} p_i' \qquad k = 1, \cdots, 6. \qquad (4)$$

On subtracting Eq. (1) from Eq. (4), there results

$$\sum_{i=1}^{3}\left(q_i' - \frac{\partial H}{\partial p_i}\right)\frac{\partial p_i}{\partial \alpha_k} - \sum_{i=1}^{3}\left(p_i' + \frac{\partial H}{\partial q_i}\right)\frac{\partial q_i}{\partial \alpha_k} = 0 \qquad k = 1, \cdots, 6. \quad (5)$$

The equations in Eq. (5) are linear and homogeneous in $(q_i' - \partial H/\partial p_i)$ and $(p_i' + \partial H/\partial q_i)$. The determinant is the functional determinant of the p_i and q_i with respect to the α's, and is not zero. Therefore,

$$q_i' = \frac{\partial H}{\partial p_i}, \qquad p_i' = -\frac{\partial H}{\partial q_i}, \qquad i = 1, 2, 3,$$

which are the equations in Eq. (367.7).

It follows that Eqs. (367.7) and (5) are equivalent since each implies the other.

371. Contact Transformations.—If a transformation is made from the variables p_i and q_i to a new set of variables P_i and Q_i, and if the two sets of variables satisfy a relation of the form

$$\sum_{i=1}^{3} q_i dp_i - \sum_{i=1}^{3} Q_i dP_i = dS,$$

where dS is an exact differential, the transformation is called a *contact transformation.*

Suppose, for example,

$$Q_i = p_i, \qquad P_i = -q_i \qquad i = 1, 2, 3.$$

Then

$$\begin{aligned}\sum q_i dp_i - \sum Q_i dP_i &= -\sum P_i dQ_i - \sum Q_i dP_i \\ &= -d\left(\sum P_i Q_i\right)\end{aligned}$$

which is an exact differential.

Suppose again that

$$\begin{aligned} q_1 &= \sqrt{2Q_1}\cos P_1, & q_2 &= Q_2, & q_3 &= Q_3; \\ p_1 &= \sqrt{2Q_1}\sin P_1, & p_2 &= P_2, & p_3 &= P_3. \end{aligned}$$

Then

$$\begin{aligned} q_1 dp_1 &= 2Q_1 \cos^2 P_1 dP_1 + \sin P_1 \cos P_1 dQ_1, \\ q_1 dp_1 - Q_1 dP_1 &= Q_1 \cos 2P_1 dP_1 + \frac{1}{2}\sin 2P_1 dQ_1, \\ &= d\left(\frac{1}{2}Q_1 \sin 2P_1\right). \end{aligned}$$

Therefore,

$$\sum(q_i dp_i - Q_i dP_i) = d\left(\frac{1}{2}Q_1 \sin P_1\right)$$

which is exact.

As a last example, let

$$\begin{aligned} Q_i &= a_{i1}q_1 + a_{i2}q_2 + a_{i3}q_3, \\ P_i &= b_{i1}p_1 + b_{i2}p_2 + b_{i3}q_3, \qquad i = 1, 2, 3, \end{aligned}$$

where the coefficients a_{ij}, b_{ij} are constants. Let

$$\Delta = |a_{ij}|, \qquad \Delta_{ij} = \text{minor of } a_{ij} \text{ in } \Delta.$$

Suppose further that the b_{ij} are related to the a_{ij} in such a way that

$$\sum P_i Q_i = \sum p_i q_i,$$

which will be the case if

$$b_{ij} = \frac{\Delta_{ij}}{\Delta}.$$

Under these hypotheses it is evident that

$$\sum q_i dp_i - \sum Q_i dP_i = 0,$$

which also is an exact differential.

372. Contact Transformations Leave the Canonical Form of the Differential Equations Unaltered.—Suppose the transformation from the variables p_i and q_i to the variables P_i and Q_i does not depend upon the time and is a contact transformation, that is,

$$\sum q_i dp_i - \sum Q_i dP_i = dS \tag{1}$$

is an exact differential, and that the p_i and q_i satisfy the canonical equations

$$q_i' = \frac{\partial H}{\partial p_i}, \qquad p_i' = -\frac{\partial H}{\partial q_i}. \tag{2}$$

From Eq. (1) it follows that

$$\sum q_i p_i' - \sum Q_i P_i' = \frac{dS}{dt},$$

and also

$$\sum q_i \frac{\partial p_i}{\partial \alpha_k} - \sum Q_i \frac{\partial P_i}{\partial \alpha_k} = \frac{\partial S}{\partial \alpha_k}.$$

Let the first of these equations be differentiated with respect to α_k and the second with respect to t. The right members being identical, the left members are equal. Hence,

$$\frac{d}{dt}\sum q_i \frac{\partial p_i}{\partial \alpha_k} - \frac{\partial}{\partial \alpha_k}\sum q_i p_i' = \frac{d}{dt}\sum Q_i \frac{\partial P_i}{\partial \alpha_k} - \frac{\partial}{\partial \alpha_k}\sum Q_i P_i'. \tag{3}$$

By Eq. (370.3), the left member of Eq. (3) is equal to $\partial H/\partial \alpha_k$. Hence,

$$\frac{d}{dt}\sum Q_i \frac{\partial P_i}{\partial \alpha_k} - \frac{\partial}{\partial \alpha_k}\sum Q_i P_i' = \frac{\partial H}{\partial \alpha_k}.$$

But this is merely Eq. (370.3) in the new variables. It follows from Sec. 370 that

$$Q_i' = \frac{\partial H}{\partial P_i}, \qquad P_i' = -\frac{\partial H}{\partial Q_i},$$

and the change of variables is canonical. The only change that is necessary is to transform H from the variables p_i and q_i to the variables P_i and Q_i.

If the time occurs explicitly in the transformation, and if Eq. (1) still holds, the time also being regarded as a variable, then the transformation is canonical with the same Hamiltonian function. But if Eq. (1) holds only with the time regarded as a constant, the transformation is canonical, but the Hamiltonian function is altered.

Suppose $F(q_1, q_2, q_3; P_1, P_2, P_3)$ is any function of the six variables $q_1, q_2, q_3; P_1, P_2, P_3$. Let there be a transformation of variables which is defined by the relations

$$p_i = \frac{\partial F}{\partial q_i}, \qquad Q_i = \frac{\partial F}{\partial P_i}; \qquad i = 1, 2, 3.$$

Then

$$\begin{aligned}\sum q_i dp_i - \sum Q_i dP_i &= \sum q_i dp_i + \sum p_i dq_i - \sum p_i dq_i - \sum Q_i dP_i \\ &= d\left(\sum p_i q_i\right) - dF;\end{aligned}$$

for

$$\begin{aligned}dF &= \sum \frac{\partial F}{\partial q_i} dq_i + \sum \frac{\partial F}{\partial P_i} dP_i, \\ &= \sum p_i dq_i + \sum Q_i dP_i.\end{aligned}$$

Hence,

$$\sum q_i dp_i - \sum Q_i dP_i = d\left[\sum p_i q_i - F\right]$$

is an exact differential, and the change of variables is canonical, a theorem which is due to Jacobi.

373. Hamilton's Theorem.—Suppose that the equations in Eq. (367.7)

$$\left.\begin{aligned} q_i' = \frac{\partial H}{\partial p_i}, \qquad p_i' &= -\frac{\partial H}{\partial q_i}, \qquad i = 1, 2, 3, \\ H &= T_2 - T_0 - U, \end{aligned}\right\} \tag{1}$$

have been completely integrated, and that

$$q_i = q_i(t;\, c_1, c_2, \cdots, c_6),$$
$$p_i = p_i(t;\, c_1, c_2, \cdots, c_6),$$

where $c_1, \cdots, c_6$ are the six constants of integration, is the solution. If these values of p_i and q_i are substituted in the expression for H it becomes a function of t and the six constants. That is,

$$H = H(t;\, c_1, c_2, \cdots, c_6).$$

Let c be any one of these constants of integration. Then

$$\begin{aligned}
\frac{\partial H}{\partial c} &= \sum_{i=1}^{3}\frac{\partial H}{\partial p_i}\frac{\partial p_i}{\partial c} + \sum_{i=1}^{3}\frac{\partial H}{\partial q_i}\frac{\partial q_i}{\partial c}, \\
&= \sum q_i'\frac{\partial p_i}{\partial c} - \sum p_i'\frac{\partial q_i}{\partial c}, && \text{by Eq. (1)}, \\
&= \frac{\partial}{\partial c}\sum p_i q_i' - \frac{d}{dt}\sum p_i\frac{\partial q_i}{\partial c} && \text{(identity)}, \\
&= \frac{\partial}{\partial c}(2T_2 + T_1) - \frac{d}{dt}\sum p_i\frac{\partial q_i}{\partial c}, && \text{by Eq. (367.2)}.
\end{aligned}$$

Hence, on replacing the value of H from Eq. (1),

$$\frac{\partial}{\partial c}(T + U) = \frac{d}{dt}\sum p_i\frac{\partial q_i}{\partial c};$$

and therefore

$$\frac{\partial}{\partial c}\int_{t_0}^{t}(T + U)dt = \sum p_i\frac{\partial q_i}{\partial c} - \sum p_{i0}\frac{\partial q_{i0}}{\partial c}, \tag{2}$$

where p_{i0} and q_{i0} are the values of p_i and q_i for $t = t_0$.

Hamilton called the integral

$$S = \int_{t_0}^{t}(T + U)dt \tag{3}$$

the *principal function.* According to its definition it is a function of t, and the six constants of integration. Let these constants be given arbitrary, but infinitesimal, increments. These increments are called *variations* and, in accordance with the notation of the calculus of variations, are denoted by the symbol δ. The variations of S and the q_k due to the variations of the constants are given by the equations

$$\delta S = \sum_{j=1}^{6}\frac{\partial S}{\partial c_j}\delta c_j \qquad \delta q_k = \sum_{j=1}^{6}\frac{\partial q_k}{\partial c_j}\delta c_j. \tag{4}$$

Let the letter c in Eq. (2) be given the subscript k; then let the equation be multiplied through by δc_k. On giving k the values $1, \cdots, 6$ in succession, six equations are thus derived from Eq. (2). Adding these six equations, there is obtained, in view of Eq. (4),

$$\delta S = \sum p_i \delta q_i - \sum p_{i0} \delta q_{i0}. \tag{5}$$

Now

$$\begin{aligned} q_i &= q_i(t;\, c_1, c_2, \cdots, c_6), \\ q_{i0} &= q_i(t_0;\, c_1, c_2, \cdots, c_6), \qquad i = 1, 2, 3. \\ S &= S(t;\, c_1, c_2, \cdots, c_6). \end{aligned}$$

Imagine the first six of these equations solved for $c_1, \cdots, c_6$ and the results substituted in the seventh. The result is

$$S = S(t;\, q_1, q_2, q_3;\, q_{10}, q_{20}, q_{30}). \tag{6}$$

If variations are given to the q_i and q_{i0} in this expression, the variation of S takes the form (t and t_0 are not varied)

$$\delta S = \sum \frac{\partial S}{\partial q_i} \delta q_i + \sum \frac{\partial S}{\partial q_{i0}} \delta q_{i0}. \tag{7}$$

A comparison of Eqs. (5) and (7) shows that, since the variations are arbitrary,

$$\frac{\partial S}{\partial q_i} = p_i, \qquad \frac{\partial S}{\partial q_{i0}} = -p_{i0}, \qquad i = 1, 2, 3. \tag{8}$$

If the constants $c_1, \cdots, c_6$ are imagined replaced by their values in terms of $q_{10}, q_{20}, q_{30};\, p_{10}, p_{20}, p_{30}$, it is evident that the equations in Eq. (8) form a complete system of integrals of the equations in Eq. (1); for the six equations in Eq. (8) could be solved for $p_1, p_2, p_3;\, q_1, q_2, q_3$ in terms of t and the six initial values.

From the manner in which the function has been derived, it would seem that the problem must first be solved before the function S can be found. There is another procedure, however, which shows that S satisfies a certain partial differential equation of the first order, as was discovered by Hamilton.

On differentiating Eq. (6) totally with respect to the time, there results

$$\begin{aligned} \frac{dS}{dt} &= \frac{\partial S}{\partial t} + \sum \frac{\partial S}{\partial q_i} q_i', \\ &= \frac{\partial S}{\partial t} + \Sigma p_i q_i', \qquad \text{by Eq. (8)}, \\ &= \frac{\partial S}{\partial t} + 2T_2 + T_1, \qquad \text{by Eq. (367.2)}. \end{aligned}$$

But, from Eq. (3),

$$\frac{dS}{dt} = T + U = T_2 + T_1 + T_0 + U.$$

On taking the difference between these two expressions, there results

$$\frac{\partial S}{\partial t} + T_2 - T_0 - U = 0;$$

or, by Eq. (367.6),

$$\frac{\partial S}{\partial t} + H(t;\ q_1, q_2, q_3;\ p_1, p_2, p_3) = 0. \tag{9}$$

The Hamiltonian function depends upon the letters p_i as well as the letters q_i; but the p_i can be replaced by $\partial S/\partial q_i$ from Eq. (8). If this is done, Eq. (9) becomes a partial differential equation of the first order and second degree, namely,

$$H\left(t;\ q_1, q_2, q_3;\ \frac{\partial S}{\partial q_1}, \frac{dS}{\partial q_2}, \frac{\partial S}{\partial q_3}\right) + \frac{\partial S}{\partial t} = 0. \tag{10}$$

Since S is a function of the four variables t; q_1, q_2, q_3, and three arbitrary constants q_{10}, q_{20}, q_{30}; it is a complete integral of the partial differential Eq. (10).

374. Jacobi's Theorem.—Hamilton discovered what is undoubtedly a remarkable theorem, but Jacobi improved it by proving the converse. If any complete integral whatever of the partial differential Eq. (373.10) can be found, a complete set of integrals of Eq. (373.1) can be derived from it; and thereby the problem is completely solved. By a complete integral of Eq. (373.10) is meant a function S of the variables t; q_1, q_2, q_3, and three arbitrary constants α_1, α_2, and α_3 in addition to a fourth constant which is always directly additive to S, since S occurs in Eq. (373.10) only through its partial derivatives, which substituted in Eq. (373.10) reduces it to an identity.

Let

$$S = S(t;\ q_1, q_2, q_3;\ \alpha_1, \alpha_2, \alpha_3) \tag{1}$$

be any complete integral of Eq. (373.10), as just defined; and let

$$\frac{\partial S}{\partial q_i} = p_i, \qquad \frac{\partial S}{\partial \alpha_i} = \beta_i, \qquad i = 1, 2, 3, \tag{2}$$

where the β_i are three new arbitrary constants. If the second set of equations in Eq. (2) are differentiated totally with respect to the time, there results

$$\frac{\partial^2 S}{\partial \alpha_i \partial t} + \frac{\partial^2 S}{\partial \alpha_i \partial q_1} q_1' + \frac{\partial^2 S}{\partial \alpha_i \partial q_2} q_2' + \frac{\partial^2 S}{\partial \alpha_i \partial q_3} q_3' = 0, \qquad i = 1, 2, 3. \tag{3}$$

On the other hand, Eq. (373.9) can be differentiated with respect to the constant α_i. If the p_i are replaced by $\partial S/\partial q_i$, Eq. (373.9) becomes an identity in t; q_1, q_2, q_3; $\alpha_1, \alpha_2, \alpha_3$. Since the constants α_1, α_2, and α_3 occur in H only as they enter through p_1, p_2, p_3, it results that

$$\frac{\partial^2 S}{\partial\alpha_i\partial t} + \frac{\partial H}{\partial p_1}\frac{\partial p_1}{\partial\alpha_i} + \frac{\partial H}{\partial p_2}\frac{\partial p_2}{\partial\alpha_i} + \frac{\partial H}{\partial p_3}\frac{\partial p_3}{\partial\alpha_i} = 0. \tag{4}$$

But from the first set of equations (Eq. (2))

$$\frac{\partial p_1}{\partial\alpha_i} = \frac{\partial^2 S}{\partial q_1\partial\alpha_i}, \qquad \frac{\partial p_2}{\partial\alpha_i} = \frac{\partial^2 S}{\partial q_2\partial\alpha_i}, \qquad \frac{\partial p_3}{\partial\alpha_i} = \frac{\partial^2 S}{\partial q_3\partial\alpha_i};$$

therefore, Eq. (4) becomes

$$\frac{\partial^2 S}{\partial\alpha_i\partial t} + \frac{\partial^2 S}{\partial\alpha_i\partial q_1}\frac{\partial H}{\partial p_1} + \frac{\partial^2 S}{\partial\alpha_i\partial q_2}\frac{\partial H}{\partial p_2} + \frac{\partial^2 S}{\partial\alpha_i\partial q_3}\frac{\partial H}{\partial p_3} = 0. \tag{5}$$

On subtracting Eq. (5) from Eq. (3), and then giving the index i the values 1, 2, 3 in succession, the following three equations result:

$$\left.\begin{aligned}
\frac{\partial^2 S}{\partial\alpha_1\partial q_1}\left(q_1' - \frac{\partial H}{\partial p_1}\right) + \frac{\partial^2 S}{\partial\alpha_1\partial q_2}\left(q_2' - \frac{\partial H}{\partial p_2}\right) + \frac{\partial^2 S}{\partial\alpha_1\partial q_3}\left(q_3' - \frac{\partial H}{\partial p_3}\right) &= 0,\\
\frac{\partial^2 S}{\partial\alpha_2\partial q_1}\left(q_1' - \frac{\partial H}{\partial p_1}\right) + \frac{\partial^2 S}{\partial\alpha_2\partial q_2}\left(q_2' - \frac{\partial H}{\partial p_2}\right) + \frac{\partial^2 S}{\partial\alpha_2\partial q_3}\left(q_3' - \frac{\partial H}{\partial p_3}\right) &= 0,\\
\frac{\partial^2 S}{\partial\alpha_3\partial q_1}\left(q_1' - \frac{\partial H}{\partial p_1}\right) + \frac{\partial^2 S}{\partial\alpha_3\partial q_2}\left(q_2' - \frac{\partial H}{\partial p_2}\right) + \frac{\partial^2 S}{\partial\alpha_3\partial q_3}\left(q_3' - \frac{\partial H}{\partial p_3}\right) &= 0.
\end{aligned}\right\} \tag{6}$$

These equations are linear and homogeneous in the quantities within the parentheses. The determinant is the functional determinant of the expressions $\frac{\partial S}{\partial\alpha_1}$, $\frac{\partial S}{\partial\alpha_2}$, and $\frac{\partial S}{\partial\alpha_3}$ with respect to q_1, q_2, q_3. If it were zero there would exist a relation between the expressions $\frac{\partial S}{\partial\alpha_1}$, $\frac{\partial S}{\partial\alpha_2}$, and $\frac{\partial S}{\partial\alpha_3}$. The three constants α_1, α_2, and α_3 would not be essentially distinct and S would not be a complete integral, which is contrary to the hypothesis. Since the determinant is not zero, each of the parentheses must be zero and, therefore,

$$q_1' = \frac{\partial H}{\partial p_1}, \qquad q_2' = \frac{\partial H}{\partial p_2}, \qquad q_3' = \frac{\partial H}{\partial p_3}. \tag{7}$$

The first set of equations (Eq. (373.1)), therefore, is satisfied.

Now let the first set of equations (2) be differentiated with respect to the time. Then

$$\frac{dp_i}{dt} = \frac{\partial^2 S}{\partial q_i \partial t} + \frac{\partial^2 S}{\partial q_i \partial q_1} q_1' + \frac{\partial^2 S}{\partial q_i \partial q_2} q_2' + \frac{\partial^2 S}{\partial q_i \partial q_3} q_3',$$

which, in view of Eq. (7), can be written

$$\frac{dp_i}{dt} = \frac{d^2 S}{\partial q_i \partial t} + \frac{\partial^2 S}{\partial q_i \partial q_1} \cdot \frac{\partial H}{\partial p_1} + \frac{\partial^2 S}{\partial q_i \partial q_2} \cdot \frac{\partial H}{\partial p_2} + \frac{\partial^2 S}{\partial q_i \partial q_3} \cdot \frac{\partial H}{\partial p_3}. \tag{8}$$

On differentiating Eq. (373.9) with respect to q_i, and bearing in mind that the p's are functions of the q's, there results

$$0 = \frac{\partial^2 S}{\partial q_i \partial t} + \frac{\partial H}{\partial q_i} + \frac{\partial H}{\partial p_1} \cdot \frac{\partial p_1}{\partial q_i} + \frac{\partial H}{\partial p_2} \cdot \frac{\partial p_2}{\partial q_i} + \frac{\partial H}{\partial p_3} \cdot \frac{\partial p_3}{\partial q_i}. \tag{9}$$

From the first equation of Eq. (374.2) there is obtained

$$\frac{\partial p_1}{\partial q_i} = \frac{\partial^2 S}{\partial q_i \partial q_1}, \qquad \frac{\partial p_2}{\partial q_i} = \frac{\partial^2 S}{\partial q_i \partial q_2}, \qquad \frac{\partial p_3}{\partial q_i} = \frac{\partial^2 S}{\partial q_i \partial q_3},$$

so that Eq. (9) can be written

$$0 = \frac{\partial^2 S}{\partial q_i \partial t} + \frac{\partial H}{\partial q_i} + \frac{\partial^2 S}{\partial q_i \partial q_1} \cdot \frac{\partial H}{\partial p_1} + \frac{\partial^2 S}{\partial q_i \partial q_2} \cdot \frac{\partial H}{\partial p_2} + \frac{\partial^2 S}{\partial q_i \partial q_3} \cdot \frac{\partial H}{\partial p_3}; \tag{10}$$

and on subtracting Eq. (10) from Eq. (8), there remains

$$\frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}, \qquad i = 1, 2, 3.$$

The second set of equations (Eq. (373.1)), therefore, is satisfied, and the equations in Eq. (374.2) are the general integrals of the given equations. If the second set of equations (Eq. 374.2),

$$\frac{\partial S}{\partial \alpha_i} = \beta_i, \qquad i = 1, 2, 3,$$

which contain only the letters $q_1, q_2, q_3; \alpha_1, \alpha_2, \alpha_3; \beta_1, \beta_2, \beta_3; t$ is solved for q_1, q_2, q_3, a complete solution of the differential equations, instead of a set of integrals, will be obtained.

375. The Restricted Case.—If the time does not occur explicitly in the Hamiltonian function, it is sufficient to take instead of Eq. (373.10),

$$H\left(q_1, q_2, q_3; \frac{\partial S}{\partial q_1}, \frac{\partial S}{\partial q_2}, \frac{\partial S}{\partial q_3}\right) = \text{constant}, \tag{1}$$

which would be the result of taking

$$S = -\alpha_3 \times t + S_1(q_1, q_2, q_3; \alpha_1, \alpha_2)$$

in Eq. (373.10), and then dropping the subscript on S_1. The process can, however, be carried out in another manner, as was done by Poincaré.

Suppose a complete solution of Eq. (1),

$$S = S(q_1, q_2, q_3; \alpha_1, \alpha_2, \alpha_3),$$

has been found. The substitution of this expression for S in Eq. (1) reduces the left member to a constant which will be a function of $\alpha_1, \alpha_2, \alpha_3$. Hence,

$$H\left(q_1, q_2, q_3; \frac{\partial S}{\partial q_1}, \frac{\partial S}{\partial q_2}, \frac{\partial S}{\partial q_3}\right) = \varphi(\alpha_1, \alpha_2, \alpha_3). \tag{2}$$

Regarding S as a function of the six variables q_i, α_i, the differential of S is

$$dS = \sum \frac{\partial S}{\partial q_i} dq_i + \sum \frac{\partial S}{\partial \alpha_i} d\alpha_i.$$

As before, take

$$\frac{\partial S}{\partial q_i} = p_i, \qquad \frac{\partial S}{\partial \alpha_i} = \beta_i, \qquad i = 1, 2, 3, \tag{3}$$

without, however, making any hypotheses as to the nature of the β's. Equations (3) furnish six relations between the twelve letters p_i, q_i, α_i, β_i. They can be regarded, therefore, as equations of transformation from the letters p_i, q_i to the letters α_i, β_i.

Since

$$dS = \sum p_i dq_i + \sum \beta_i d\alpha_i,$$

the relation

$$\sum \beta_i d\alpha_i - \sum q_i dp_i = d\left(S - \sum p_i q_i\right)$$

is an exact differential, and therefore, by Sec. 372, the transformation of variables is canonical. Consequently,

$$\frac{d\beta_i}{dt} = \frac{\partial H}{\partial \alpha_i}, \qquad \frac{d\alpha_i}{dt} = -\frac{\partial H}{\partial \beta_i}. \tag{4}$$

By Eq. (2), the transformed value of H is $\varphi(\alpha_1, \alpha_2, \alpha_3)$, which is independent of $\beta_1, \beta_2, \beta_3$. Hence,

$$\frac{d\beta_i}{dt} = \frac{\partial \varphi}{\partial \alpha_i}, \qquad \frac{d\alpha_i}{dt} = 0; \tag{5}$$

and these equations are integrable at once, for

$$\alpha_i = \text{const}, \qquad i = 1, 2, 3. \tag{6}$$

Therefore, $\varphi(\alpha_1, \alpha_2, \alpha_3)$ is a constant, and likewise $\dfrac{\partial \varphi}{\partial \alpha_i}$, so that

$$\beta_i = \frac{\partial \varphi}{\partial \alpha_i} t + \beta_{i0}, \qquad i = 1, 2, 3, \tag{7}$$

where the β_{i0} are three new constants of integration.

The solution of the three equations

$$\frac{\partial S}{\partial \alpha_i} = \frac{\partial \varphi}{\partial \alpha_i} t + \beta_{i0}, \qquad i = 1, 2, 3,$$

will give q_1, q_2, and q_3 as functions of t and the six constants α_1, α_2, α_3; β_{10}, β_{20}, β_{30}.

376. The Trajectories are Perpendicular to the Surface $S = C$. Two displacements of a point,

$$dx,\ dy,\ dz \qquad \text{and} \qquad \delta x,\ \delta y,\ \delta z,$$

in rectangular coordinates, will be mutually perpendicular if, and only if,

$$dx\delta x + dy\delta y + dz\delta z = 0.$$

On dividing this expression through by dt, it is seen that the displacement δx, δy, δz will be perpendicular to the velocity if, and only if,

$$x'\delta x + y'\delta y + z'\delta z = 0. \tag{1}$$

In the generalized coordinates.

$$x = \varphi_1(q_1, q_2, q_3), \qquad y = \varphi_2(q_1, q_2, q_3), \qquad z = \varphi_3(q_1, q_2, q_3);$$

$$\left.\begin{aligned} x' &= \frac{\partial \varphi_1}{\partial q_1} q_1' + \frac{\partial \varphi_1}{\partial q_2} q_2' + \frac{\partial \varphi_1}{\partial q_3} q_3', \qquad \cdots, \qquad \cdots, \\ \delta x &= \frac{\partial \varphi_1}{\partial q_1} \delta q_1 + \frac{\partial \varphi_1}{\partial q_2} \delta q_2 + \frac{\partial \varphi_1}{\partial q_3} \delta q_3, \qquad \cdots, \qquad \cdots; \end{aligned}\right\} \tag{2}$$

and

$$\begin{aligned} T = \frac{m}{2}\Bigg[&\left(\frac{\partial \varphi_1}{\partial q_1} q_1' + \frac{\partial \varphi_1}{\partial q_2} q_2' + \frac{\partial \varphi_1}{\partial q_3} q_3'\right)^2 \\ &+ \left(\frac{\partial \varphi_2}{\partial q_1} q_1' + \frac{\partial \varphi_2}{\partial q_2} q_2' + \frac{\partial \varphi_2}{\partial q_3} q_3'\right)^2 \\ &+ \left(\frac{\partial \varphi_3}{\partial q_1} q_1' + \frac{\partial \varphi_3}{\partial q_2} q_2' + \frac{\partial \varphi_3}{\partial q_3} q_3'\right)^2 \Bigg]. \end{aligned}$$

If the values of x', y', z'; δx, δy, δz are substituted from Eq. (2) in Eq. (1), it will be found that the condition for orthogonality (Eq. (1)) can be written

$$\frac{\partial T}{\partial q_1'} \delta q_1 + \frac{\partial T}{\partial q_2'} \delta q_2 + \frac{\partial T}{\partial q_3'} \delta q_3 = 0,$$

or

$$p_1 \delta q_1 + p_2 \delta q_2 + p_3 \delta q_3 = 0. \tag{3}$$

Now consider any displacement of a point along the surface

$$S(q_1, q_2, q_3; \alpha_1, \alpha_2, \alpha_3) = \text{constant}.$$

It will satisfy the equation

$$\frac{\partial S}{\partial q_1}\delta q_1 + \frac{\partial S}{\partial q_2}\delta q_2 + \frac{\partial S}{\partial q_3}\delta q_3 = 0,$$

and therefore

$$p_1\delta q_1 + p_2\delta q_2 + p_3\delta q_3 = 0;$$

that is, the displacement is normal to the velocity. Hence, if a trajectory for which α_1, α_2, and α_3 are constants passes through a point q_1, q_2, q_3, the surface S which passes through the same point is normal to the trajectory.

377. Plane Motion under a Central Force.—For a unit particle, the expression for the kinetic energy in polar coordinates for motion in a plane is

$$T = \frac{1}{2}(r'^2 + r^2\theta'^2);$$

and if the force is a function of the distance only, the potential function is

$$U = f(r),$$

the function $f(r)$ depending upon the law of force which will be left undetermined.

On taking

$$r = q_1, \qquad \theta = q_2,$$

$$r_1' = p_1, \qquad \theta' = \frac{p_2}{q_1^2},$$

the Hamiltonian function is

$$T - U = H = \frac{1}{2}p_1^2 + \frac{1}{2}\frac{p_2^2}{q_1^2} - f(q_1).$$

The partial differential equation of Jacobi is obtained from H by setting

$$p_1 = \frac{\partial S}{\partial q_1}, \qquad p_2 = \frac{\partial S}{\partial q_2}$$

in H and then equating the result to a constant. The result is

$$\frac{1}{2}\left(\frac{\partial S}{\partial q_1}\right)^2 + \frac{1}{2q_1^2}\left(\frac{\partial S}{\partial q_2}\right)^2 - f(q_1) = C = \text{constant}.$$

It will be observed that this equation involves q_2 only through its partial derivative. For this reason, let

$$S = S_1 + \alpha_2 q_2,$$

where S_1 is a function of q_1 alone, and α_2 is a constant. The partial differential equation then reduces to the ordinary differential equation

$$\left(\frac{dS_1}{dq_1}\right)^2 + \frac{\alpha_2^2}{q_1^2} - 2f(q_1) = 2C;$$

and therefore

$$\frac{dS_1}{dq_1} = \pm\sqrt{2C + 2f(r) - \frac{\alpha_2^2}{q_1^2}}.$$

If α_1 is the value of q_1 for which dS_1/dq_1 vanishes, then

$$C = \frac{\alpha_2^2}{2\alpha_1^2} - f(\alpha_1), \tag{1}$$

$$S_1 = \int_{\alpha_1}^{q_1}\sqrt{\left(\frac{1}{\alpha_1^2} - \frac{1}{q_1^2}\right)\alpha_2^2 + 2(f(q_1) - f(\alpha_1))} \cdot dq_1,$$

and

$$S = \alpha_2 q_2 + \int_{\alpha_1}^{q_1}\sqrt{\left(\frac{1}{\alpha_1^2} - \frac{1}{q_1^2}\right)\alpha_2^2 + 2(f(q_1) - f(\alpha_1))} \cdot dq_1. \tag{2}$$

From this expression for S, it follows that:

$$\begin{aligned}
\frac{\partial S}{\partial \alpha_1} &= \beta_1 = -\left[\frac{\alpha_2^2}{\alpha_1^3} + \frac{df(\alpha_1)}{d\alpha_1}\right]\int_{\alpha_1}^{q_1}\frac{dq_1}{\sqrt{\left(\frac{1}{\alpha_1^2} - \frac{1}{q_1^2}\right)\alpha_2^2 + 2(f(q_1) - f(\alpha_1))}},\\
\frac{\partial S}{\partial \alpha_2} &= \beta_2 = q_2 + \int_{\alpha_1}^{q_1}\frac{\alpha_2\left(\frac{1}{\alpha_1^2} - \frac{1}{q_1^2}\right)dq_1}{\sqrt{\left(\frac{1}{\alpha_1^2} - \frac{1}{q_1^2}\right) + 2(f(q_1) - f(\alpha_1))}}.
\end{aligned} \tag{3}$$

The new variables satisfy the differential equations

$$\frac{d\beta_1}{dt} = \frac{\partial C}{\partial \alpha_1}, \qquad \frac{d\alpha_1}{dt} = -\frac{\partial C}{\partial \beta_1},$$
$$\frac{d\beta_2}{dt} = \frac{\partial C}{\partial \alpha_2}, \qquad \frac{d\alpha_2}{dt} = -\frac{\partial C}{\partial \beta_2},$$

since in these variables H is equal to C. But inasmuch as C (Eq. (1)) does not depend upon β_1 nor β_2, both α_1 and α_2 are constants. The equations for β_1 and β_2 give

$$\beta_1 = -\left[\frac{\alpha_2^2}{\alpha_1^3} + \frac{df(\alpha_1)}{d\alpha_1}\right](t - t_0), \qquad \beta_2 = \frac{\alpha_2}{\alpha_1^2}(t + t_2), \tag{4}$$

where t_0 and t_2 are the two constants of integration.

On equating the two values of β_1 (Eqs. (3) and (4)), it is found that

$$t - t_0 = \pm \int_{\alpha_1}^{q_1} \frac{dq_1}{\sqrt{\left(\frac{1}{\alpha_1^2} - \frac{1}{q_1^2}\right)\alpha_2^2 + 2(f(q_1) - f(\alpha_1))}}. \tag{5}$$

If this expression for t is multiplied by α_2/α_1^2, the left member differs from β_2 only by an arbitrary constant. Hence, by subtraction from the second of Eq. (3), there results

$$q_2 = \pm \int_{\alpha_1}^{q_1} \frac{\alpha_2 dq_1}{q_1^2\sqrt{\left(\frac{1}{\alpha_1^2} - \frac{1}{q_1^2}\right)\alpha_2^2 + 2(f(q_1) - f(\alpha_1))}} + \text{constant}. \tag{6}$$

Since Eq. (6) is independent of the time, it is the equation of the orbit.

378. Simple Harmonic Motion.—If the force is attractive and directly proportional to the distance,

$$f(r) = -\frac{1}{2}k^2r^2,$$

where k^2 is the factor of proportionality. On multiplying through by k, Eq. (377.5) becomes

$$k(t - t_0) = \pm \int_{\alpha_1}^{q_1} \frac{dq_1}{\sqrt{\left(\frac{1}{\alpha_1^2} - \frac{1}{q_1^2}\right)\frac{\alpha_2^2}{k^2} + (\alpha_1^2 - q_1^2)}}$$

One of the roots of the radicand is $q_1 = \alpha_1$. Let α_2^2 be chosen so that $q_1 = \alpha$ is the other root, with $\alpha_1 > \alpha$. Then

$$\alpha_2^2 = k^2\alpha_1^2\alpha,^2 \tag{1}$$

and the negative sign must be taken before the integral, since initially q_1 equals α_1 and decreases as t increases. Therefore,

$$k(t - t_0) = -\int_{\alpha_1}^{q_1} \frac{q_1 dq_1}{\sqrt{(\alpha_1^2 - q_1^2)(q_1^2 - \alpha^2)}}.$$

Similarly, Eq. (377.6) gives

$$q_2 - q_{20} = -\int_{\alpha_1}^{q_1} \frac{\alpha\alpha_1 dq_1}{q_1\sqrt{(\alpha_1^2 - q_1^2)(q_1^2 - \alpha^2)}}.$$

These expressions are readily integrated, and give

$$\left.\begin{aligned} q_1^2 &= \alpha_1^2 \cos^2 k(t - t_0) + \alpha^2 \sin^2 k(t - t_0), \\ q_1^2 &= \frac{\alpha^2\alpha_1^2}{\alpha^2 \cos^2 (q_2 - q_{20}) + \alpha_1^2 \sin^2 (q_2 - q_{20})}. \end{aligned}\right\} \tag{2}$$

If it is remembered that q_1 is the radius vector and q_2 the polar angle, it will be seen that this last equation is the polar equation of an ellipse with the origin at the center; the major semiaxis is α_1, and the minor semiaxis is α. In the customary notation

$$q_1 = r, \qquad q_2 = \theta; \qquad \alpha_1 = a, \qquad \alpha_2 = b.$$

The curves which are orthogonal to the family of ellipses which have a common center and the same values of α_1 and α_2, that is, differing only by a rotation, are given by Eq. (377.2)

$$S = \alpha_2 q_2 - \int_{\alpha_1}^{q_1} \sqrt{\left(\frac{1}{\alpha_1^2} - \frac{1}{q_1^2}\right)\alpha_2^2 + k^2(\alpha_1^2 - q_1^2)}\, dq_1 = \text{constant}.$$

After substituting the value of α_2 from Eq. (1) and then removing a superfluous factor $k\alpha_1$, this expression reduces to

$$q_2 - q_{20} = \frac{1}{\alpha_1\alpha}\int_{\alpha_1}^{q_1} \sqrt{(\alpha_1^2 - q_1^2)(q_1^2 - \alpha^2)}\,\frac{dq_1}{q_1}. \tag{3}$$

For the purpose of integration, let

$$q_1^2 = \frac{1}{2}(\alpha_1^2 + \alpha^2) + \frac{1}{2}(\alpha_1^2 - \alpha^2)\cos 2\omega, \tag{4}$$

where ω is a new parameter. In reality, ω does not differ from $k(t - t_0)$, as is seen by comparison with Eq. (2). Then

$$q_2 - q_{20} = \frac{1}{2\alpha_1\alpha}\int_0^{\omega} \frac{(\alpha_1^2 - \alpha^2)^2(\cos^2 2\omega - 1)d\omega}{(\alpha_1^2 - \alpha^2)\cos 2\omega + (\alpha_1^2 + \alpha^2)}$$

$$= \frac{1}{2\alpha_1\alpha}\int_0^{\omega}\left[(\alpha_1^2 - \alpha^2)\cos 2\omega - (\alpha_1^2 + \alpha^2) + \frac{4\alpha_1^2\alpha^2}{(\alpha_1^2 - \alpha^2)\cos 2\omega + (\alpha_1^2 + \alpha^2)}\right]d\omega.$$

Hence,

$$q_2 - q_{20} = \frac{1}{2\alpha_1\alpha}\left[\frac{1}{2}(\alpha_1^2 - \alpha^2)\sin 2\omega - (\alpha_1^2 + \alpha^2)\omega + 2\alpha_1\alpha\tan^{-1}\left(\frac{\alpha}{\alpha_1}\tan\omega\right)\right]. \tag{5}$$

Equations (4) and (5) are the parametric equations of the orthogonal curve. If the major axis is twice the minor axis, the curve (Fig. 181) resembles an eight-leaved sunflower.

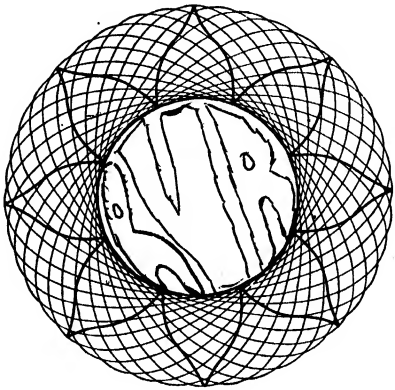


FIG. 181.

379. The Spherical Pendulum.—The spherical pendulum, which was discussed in Sec. 340, will be used as a second illustration of the use of canonical equations. In the problem of central forces (Sec. 377), the second method of Sec. 375 was employed. By way of contrast, the first method will be used in the present problem.

For the sake of simplicity of notation, let the mass of the pendulum bob and the radius of the sphere be taken equal to unity. If spherical coordinates are used, the expressions for the kinetic energy and the potential function are

$$T = \frac{1}{2}(\varphi'^2 + \theta'^2 \cos^2 \varphi), \qquad U = -g \sin \varphi;$$

so that

$$T - U = H = \frac{1}{2}\varphi'^2 + \frac{1}{2}\theta'^2 \cos^2 \varphi + g \sin \varphi.$$

For the purpose of expressing H in canonical variables, let

$$\varphi = q_1, \qquad \theta = q_2, \qquad \frac{\partial T}{\partial \varphi'} = p_1, \qquad \frac{\partial T}{\partial \theta'} = p_2;$$

and therefore

$$\varphi' = p_1, \qquad \theta' = p_2 \sec^2 q_1.$$

The expression for H, therefore, in canonical variables is

$$H = \frac{1}{2}p_1^2 + \frac{1}{2}p_2^2 \sec^2 q_1 + g \sin q_1. \tag{1}$$

On setting

$$p_1 = \frac{\partial S}{\partial q_1}, \qquad p_2 = \frac{\partial S}{\partial q_2}$$

in H, the partial differential equation of Hamilton and Jacobi (Eq. (375.1)) is found to be

$$\frac{1}{2}\left(\frac{\partial S}{\partial q_1}\right)^2 + \frac{1}{2}\left(\frac{\partial S}{\partial q_2}\right)^2 \sec^2 q_1 + g \sin q_1 = \alpha_2. \tag{2}$$

Inasmuch as the pendulum has but two degrees of freedom and one of the constants α_2 is already in evidence, it is necessary for a complete solution to find a function S of q_1 and q_2 which contains a single new arbitrary constant α_1. Since the coordinate q_2 appears in Eq. (2) only through its derivative, it is sufficient to take

$$S = \alpha_1 q_2 + S_1;$$

where α_1 is an arbitrary constant, and S_1 is a function of q_1 alone. By means of this substitution the partial differential equation reduces to the ordinary differential equation

$$\left(\frac{dS_1}{dq_1}\right)^2 + \alpha_1^2 \sec^2 q_1 + 2g \sin q_1 = 2\alpha_2, \tag{3}$$

the solution of which is

$$S_1 = \pm\int\sqrt{2\alpha_2 - 2g \sin q_1 - \alpha_1^2 \sec^2 q_1}\,dq_1.$$

The complete expression for S, then, is

$$S = -\alpha_2 t + \alpha_1 q_2 \pm \int\sqrt{2\alpha_2 - 2g \sin q_1 - \alpha_1^2 \sec^2 q_1}\,dq_1. \tag{4}$$

The expressions for p_1 and p_2 are not needed for the solution of the problem; but, since

$$\frac{\partial S}{\partial q_2} = p_2 = \alpha_1$$

is a constant, it is worthy of note that this is the integral

$$\theta' \cos^2 \varphi = h \tag{5}$$

of Sec. 340.

The solution of the problem is furnished by the two expressions

$$\left.\begin{aligned} \frac{\partial S}{\partial \alpha_1} &= \beta_1 = q_2 \pm \alpha_1 \int \frac{\sec^2 q_1\, dq_1}{\sqrt{2\alpha_2 - 2g \sin q_1 - \alpha_1^2 \sec^2 q_1}}, \\ \frac{\partial S}{\partial \alpha_2} &= \beta_2 = -t \pm \int \frac{dq_1}{\sqrt{2\alpha_2 - 2g \sin q_1 - \alpha_1^2 \sec^2 q_1}}, \end{aligned}\right\} \tag{6}$$

where β_1 and β_2 are two new constants. The first equation is independent of the time, and is therefore the equation of the path of the pendulum. The second equation then gives the time.

If it is borne in mind that

$$q_1 = \varphi, \qquad q_2 = \theta,$$

these results are easily identified with Eqs. (341.2) and (341.1).

380. The Motion of a Planet in Three Dimensions.—If spherical coordinates are used, the expressions for the potential function and the kinetic energy for the motion of a planet about the sun, referred to the sun as the origin, as derived from Secs. 293 and 308, are

$$T = \frac{1}{2}(r'^2 + r^2\varphi'^2 + r^2\cos^2\varphi\cdot\theta'^2), \qquad U = \frac{k^2}{r};$$

where φ is the latitude, θ is the longitude, and k^2 is a factor of proportionality, the mass of the planet having been taken equal to unity, or divided out, as it is preferred. Let r, φ, θ play the rôle of q_1, q_2, q_3, respectively; and let R, Φ, Θ play the rôle of p_1, p_2, p_3, respectively. Then

$$R = \frac{\partial T}{\partial r'} = r', \qquad \Phi = \frac{\partial T}{\partial \varphi'} = r^2\varphi', \qquad \Theta = \frac{\partial T}{\partial \theta'} = r^2\cos^2\varphi\cdot\theta'.$$

The expression for H then becomes

$$H = T - U = \frac{1}{2}\left(R^2 + \frac{1}{r^2}\Phi^2 + \frac{1}{r^2\cos^2\varphi}\Theta^2\right) - \frac{k^2}{r}.$$

Hence, the partial differential equation of Hamilton and Jacobi is

$$\frac{1}{2}\left[\left(\frac{\partial S}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial S}{\partial \varphi}\right)^2 + \frac{1}{r^2\cos^2\varphi}\left(\frac{\partial S}{\partial \theta}\right)^2\right] = \frac{k^2}{r} + k^2C, \tag{1}$$

with $$\frac{\partial S}{\partial r} = R, \qquad \frac{\partial S}{\partial \varphi} = \Phi, \qquad \frac{\partial S}{\partial \theta} = \Theta.$$

Again it will be observed that the differential equation does not contain θ explicitly. Let it be assumed that

$$S = S_r + S_\varphi + kc\theta,$$

where S_r is a function of r alone, S_φ is a function of φ alone, and c is an arbitrary constant. The equation then reduces to

$$\left(\frac{dS_r}{dr}\right)^2 + \frac{1}{r^2}\left[\left(\frac{dS_\varphi}{d\varphi}\right)^2 + \frac{k^2c^2}{\cos^2\varphi}\right] = \frac{2k^2}{r} + 2k^2C. \tag{2}$$

Since the variable φ occurs only in the second term of the left member, it is necessary that

$$\left(\frac{\partial S_\varphi}{\partial \varphi}\right)^2 + \frac{k^2c^2}{\cos^2 \varphi}$$

shall be a positive constant, say $k^2\alpha_2^2$; and the differential equation then reduces to an ordinary differential equation in r, namely,

$$\left(\frac{dS_r}{dr}\right)^2 = k^2\left(-\frac{\alpha_2^2}{r^2} + \frac{2}{r} + 2C\right).$$

Let α_1 be a value of r for which the derivative vanishes. Then

$$C = \frac{\alpha_2^2}{2\alpha_1^2} - \frac{1}{\alpha_1}$$

and

$$S_r = \pm k\int_{\alpha_1}^{r}\sqrt{\left(\frac{1}{\alpha_1^2} - \frac{1}{r^2}\right)\alpha_2^2 + 2\left(\frac{1}{r} - \frac{1}{\alpha_1}\right)}dr.$$

Likewise, let α_3 be the value of φ for which the derivative of S_φ vanishes. Then

$$c = \alpha_2 \cos \alpha_3$$

and

$$S_\varphi = \pm k\alpha_2\int_0^\varphi \sqrt{1 - \cos^2 \alpha_3 \sec^2 \varphi}\,d\varphi.$$

The complete solution of the partial differential equation then is

$$S = \pm k\int_{\alpha_1}^{r}\sqrt{\left(\frac{1}{\alpha_1^2} - \frac{1}{r^2}\right)\alpha_2^2 + 2\left(\frac{1}{r} - \frac{1}{\alpha_1}\right)}dr$$
$$\pm k\alpha_2\int_0^\varphi \sqrt{1 - \cos^2 \alpha_3 \sec^2 \varphi}\,d\varphi + k\theta\alpha_2\cos\alpha_3. \quad (3)$$

The new variables β_1, β_2, and β_3 are

$$\frac{\partial S}{\partial \alpha_1} = \beta_1 = \pm k\frac{\alpha_1 - \alpha_2^2}{\alpha_1^3}\int_{\alpha_{r1}}^{r}\frac{dr}{\sqrt{\left(\frac{1}{\alpha_1^2} - \frac{1}{r^2}\right)\alpha_2^2 + 2\left(\frac{1}{r} - \frac{1}{\alpha_1}\right)}}, \quad (4)$$

$$\frac{\partial S}{\partial \alpha_2} = \beta_2 = \pm k\int_{\alpha_1}\frac{\alpha_2\left(\frac{1}{\alpha_1^2} - \frac{1}{r^2}\right)dr}{\sqrt{\left(\frac{1}{\alpha_1^2} - \frac{1}{r^2}\right)\alpha_2^2 + 2\left(\frac{1}{r} - \frac{1}{\alpha_1}\right)}}$$
$$\pm k\int_0^\varphi \sqrt{1 - \cos^2 \alpha_3 \sec^2 \varphi}\,d\varphi + k\theta\cos\alpha_3,$$

$$\frac{\partial S}{\partial \alpha_3} = \beta_3 = -k\theta\alpha_2 \sin \alpha_3 \pm k\alpha_2\int_0^\varphi \frac{\sin\alpha_3\cos\alpha_3\sec^2\varphi}{\sqrt{1 - \cos^2\alpha_3\sec^2\varphi}}d\varphi.$$

Also, by Eq. (375.7),

$$\left.\begin{aligned}
\beta_1 &= k^2\frac{\partial C}{\partial \alpha_1}t + \beta_{10} = k^2\frac{\alpha_1 - \alpha_2^2}{\alpha_1^3}(t - t_0),\\
\beta_2 &= k^2\frac{\partial C}{\partial \alpha_2}t + \beta_{20} = k^2\frac{\alpha_2}{\alpha_1^2}(t - t_1),\\
\beta_3 &= k^2\frac{\partial C}{\partial \alpha_3}t + \beta_{30} = 0 + \beta_{30}.
\end{aligned}\right\} \tag{5}$$

On equating the right member of the first equation of Eq. (5) to the right member of the first equation of Eq. (4) and then removing the superfluous factor, there results

$$k(t - t_0) = \pm\int_{\alpha_1}^{r}\frac{dr}{\sqrt{\left(\frac{1}{\alpha_1^2} - \frac{1}{r^2}\right)\alpha_2^2 + 2\left(\frac{1}{r} - \frac{1}{\alpha_1}\right)}}. \tag{6}$$

The factors of the radicand in this integral are

$$\left(\frac{1}{\alpha_1} - \frac{1}{r}\right) \quad \text{and} \quad \left(\frac{1}{\alpha_1} + \frac{1}{r}\right)\alpha_2^2 - 2.$$

By its definition, the constant α_1 is necessarily positive, since r is always positive. Consequently, the second factor can vanish if, and only if, $\alpha_1 \geqq \alpha_2^2/2$. If $\alpha_1 > \alpha_2^2/2$, there is a second value of r for which the radicand vanishes, and the orbit is an ellipse. If $\alpha_1 = \alpha_2^2/2$, this second value of r recedes to infinity, and the orbit is a parabola. If $\alpha_1 < \alpha_2^2/2$, the radicand vanishes but once, and the orbit is an hyperbola. If the substitutions

$$\alpha_1 = a(1 - e), \qquad \alpha_2^2 = a(1 - e^2)$$

are made, these three conditions are reduced to the three possibilities

$$e \lesseqqgtr +1.$$

With these values of the constants, Eq. (6) becomes

$$k(t - t_0) = \frac{+1}{\sqrt{a}}\int_{a(1-e)}^{r}\frac{dr}{\sqrt{\frac{\epsilon^2}{r^2} - \left(\frac{1}{a} - \frac{1}{r}\right)^2}},$$

which is essentially the same as Eq. (294.1) and can be integrated in the same manner.

Since β_3 is a constant, it can be chosen so that, after the factor $k\alpha_2 \sin \alpha_3$ is divided out, the third equation of Eq. (4) becomes

$$\theta - \theta_0 = \int_0^{\varphi} \frac{\cos \alpha_3 \sec^2 \varphi}{\sqrt{1 - \cos^2 \alpha_3 \sec^2 \varphi}}\, d\varphi.$$

This expression is easily integrated, giving

$$\sin (\theta - \theta_0) = \frac{\tan \varphi}{\tan \alpha_3}.$$

In Fig. 182, let NPM be the intersection of the plane of the orbit on the unit sphere. Let the inclination of the orbit to the xy-

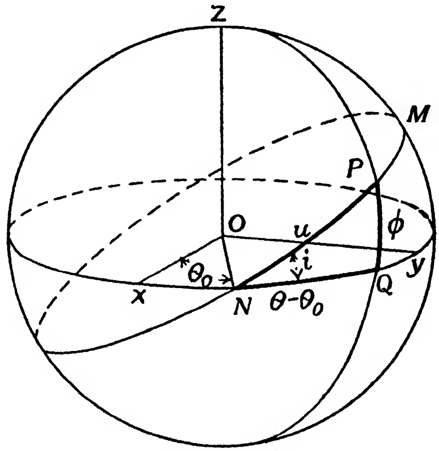


FIG. 182.

plane be i. Let P be the projection of the planet, and N the projection of the node. Then

$$\theta_0 = xN, \qquad \theta - \theta_0 = NQ.$$

In the right spherical triangle NPQ, the relations (Eq. (261.1))

$$\left.\begin{aligned} \cos u &= \cos (\theta - \theta_0) \cos \varphi, \\ \sin u \cos i &= \sin (\theta - \theta_0) \cos \varphi, \\ \sin u \sin i &= \sin \varphi, \end{aligned}\right\} \tag{7}$$

hold, and therefore,

$$\sin (\theta - \theta_0) = \frac{\tan \varphi}{\tan i}. \tag{8}$$

A comparison of these two expressions for the $\sin (\theta - \theta_0)$ shows that the constant α_3 can be identified with the inclination of the plane of the orbit. It is evident also, that θ_0 is the longitude of the node.

By means of the relation which has just been derived between φ and θ, Eq. (8), the second integral of the second equation of Eq. (4) can be expressed in terms of θ, that is,

$$\int_0^{\varphi} \sqrt{1 - \cos^2 \alpha_3 \sec^2 \varphi}\, d\varphi$$
$$= \int_{\theta_0}^{\theta} \left[\frac{\sec \alpha_3}{1 + \tan^2 \alpha_3 \sin^2 (\theta - \theta_0)} - \cos \alpha_3 \right] d\theta.$$

Therefore,

$$k\left[\int_0^{\varphi} \sqrt{1 - \cos^2 \alpha_3 \sec^2 \varphi}\, d\varphi + (\theta - \theta_0) \cos \alpha_3 \right]$$
$$= k \tan^{-1} (\sec \alpha_3 \tan (\theta - \theta_0)).$$

Since $\alpha_3 = i$, it is found from Eq. (7) that

$$\tan u = \sec \alpha_3 \tan (\theta - \theta_0).$$

Therefore,

$$k\left[\int_0^{\varphi} \sqrt{1 - \cos^2 \alpha_3 \sec^2 \varphi}\, d\varphi + (\theta - \theta_0) \cos \alpha_3 \right] = ku, \tag{9}$$

where u is measured from the node N.

On substituting Eq. (9) in the second of Eq. (4), and bearing in mind (Eq. (6)) that

$$\frac{dr}{\sqrt{\left(\frac{1}{\alpha_1^2} - \frac{1}{r^2}\right)\alpha_2^2 + \left(\frac{1}{r} - \frac{1}{\alpha_1}\right)}} = kdt,$$

the second equation of Eq. (4) reduces to

$$\beta_2 = k^2\alpha_2 \int_{t_0}^{t} \left(\frac{1}{\alpha_1^2} - \frac{1}{r^2}\right) dt + ku,$$
$$= k^2 \frac{\alpha_2}{\alpha_1^2}(t - t_0) - k^2\alpha_2 \int_{t_0}^{t} \frac{dt}{r^2} + ku.$$

But from the second of Eq. (5)

$$\beta_2 = k^2 \frac{\alpha_2}{\alpha_1^2}(t - t_1);$$

therefore,

$$u = k\alpha_2 \int_{t_0}^{t} \frac{dt}{r^2} + \text{constant}.$$

Since the integral in this expression vanishes at the perihelion point, the "constant" is the arc u measured from the node to the projection of the perihelion point upon the unit sphere. It is usually denoted by the symbol ϖ, and is called (really miscalled) the *longitude of the perihelion from the node.* It is measured in the plane of the orbit and not in the xy-plane. It is, therefore.

not a true longitude. The true anomaly v is measured in the plane of the orbit from the perihelion. Hence,

$$v = u - \varpi = k\alpha_2 \int_{t_0}^{t} \frac{dt}{r^2}.$$

This completes the solution of the problem. It will be observed that the constants α_1, α_2, and α_3 are functions of the classical elements a, e, and i; while the elements $\theta_0 = \Omega$, ϖ, and $t_0 = T$ enter only through the constants β_{10}, β_{20}, and β_{30}.

381. Variation of Parameters.—Let it be supposed that the differential equations are canonical and that

$$q_i' = \frac{\partial H}{\partial p_i}, \qquad p_i' = -\frac{\partial H}{\partial q_i}, \qquad i = 1, 2, 3. \tag{1}$$

Suppose further that H is a function of a parameter μ, which can be written

$$H = H_0 + \mu H_1,$$

where H_1 may or may not be a function of the parameter μ, but H_0 does not contain μ. Suppose, finally, that it is known how to solve the equations

$$q_i' = \frac{\partial H_0}{\partial p_i}, \qquad p_i' = -\frac{\partial H_0}{\partial q_i}, \qquad i = 1, 2, 3. \tag{2}$$

That is to say, a function $S(q_1, q_2, q_3; \alpha_1, \alpha_2, \alpha_3; t)$ can be found in which the α_1, α_2, α_3 are the three independent arbitrary constants, which satisfies the Hamilton-Jacobi differential equation

$$H_0\left(q_1, q_2, q_3; \frac{\partial S}{\partial q_1}, \frac{\partial S}{\partial q_2}, \frac{\partial S}{\partial q_3}; t\right) + \frac{\partial S}{\partial t} = 0.$$

Then the integrals of Eq. (2) are

$$p_i = \frac{\partial S}{\partial q_i}, \qquad \beta_i = \frac{\partial S}{\partial \alpha_i}, \qquad i = 1, 2, 3, \tag{3}$$

where the β_i are three new constants of integration.

The six equations (Eq. (3)) can be imagined as solved for the p_i and q_i as functions of the constants α_i, β_i, and t. That is,

$$\begin{aligned} p_i &= p_i(\alpha_1, \alpha_2, \alpha_3; \beta_1, \beta_2, \beta_3; t), \\ q_i &= q_i(\alpha_1, \alpha_2, \alpha_3; \beta_1, \beta_2, \beta_3; t); \end{aligned} \tag{4}$$

and these expressions substituted in Eq. (2) reduce the differential equations to an identity.

Equation (4) can be regarded, however, not as solutions of Eq. (2), but merely as equations of transformation of variables. Then, on account of the relations in Eq. (3) and the theorem of

Sec. 372, the transformation is a canonical one, and the differential equations are

$$\beta_i' = \mu\frac{\partial H_1}{\partial \alpha_i}, \qquad \alpha_i' = -\mu\frac{\partial H_1}{\partial \alpha_i}, \qquad i = 1, 2, 3. \tag{5}$$

The term H_0 disappears from the Hamiltonian function; as is evident from the fact that for μ equal to zero the α_i and β_i are constants.

This is the expression in canonical variables of the method of the variation of arbitrary parameters which has played such an important rôle in the theory of the planetary perturbations. It is evident that if the right members of Eq. (5) are very small, the α_i and β_i, although functions of the time, will vary very slowly, and will continue to vary slowly as long as the right members remain small. These conditions are satisfied in the astronomical case just mentioned, so that the elements of an orbit are not actually constants, on account of the perturbations of the planets; but, in general, they vary so slowly that for many purposes they can be regarded as constants.

Problems XXIV

1. Show that the change of variables from p_1, q_1 to P_1, Q_1 by means of the relations

$$q_1 = \varphi_1(P_1, Q_1), \qquad p_1 = \varphi_2(P_1, Q_1)$$

is canonical if the functional determinant

$$\frac{\partial(q_1, p_1)}{\partial(Q_1, P_1)} = 1.$$

2. If the transformation of variables is linear with constant coefficients (Sec. 371) and if it is an orthogonal transformation, that is,

$$\sum_{j=1}^{3} a_{ij}^2 = 1, \qquad \sum_{j=1}^{3} a_{ij}a_{kj} = 0,$$

the transformation of variables is canonical.

3. Prove in detail, if the problem is two dimensional, that the equation $S(q_1, q_2, \alpha_1, \alpha_2) =$ const. represents curves which are orthogonal to the trajectories.

4. Solve the problems of central forces and planetary motion by the first method of Sec. 375.

5. Solve the problem of the spherical pendulum by the second method.

6. Solve the problem of a projectile *in vacuo*, and determine the orthogonal curves.

7. Solve the problem of the simple pendulum by means of canonical equations.

8. Write and solve the canonical equations for the problem of Sec. 356.

CHAPTER XVI

THE GENERAL PRINCIPLES OF MECHANICS

382. The Foundations of Mechanics.—All of the developments and theorems of the preceding chapters have been built upon Newton's three laws of motion as a foundation. These three laws, together with one or two subsidiary propositions, such as that of the transmissibility of force, have sufficed. Since the time of Newton, mathematicians have shown a strong desire to find a *single* principle from which all of the theorems of mechanics could be drawn, and four of such principles have been developed. By combining d'Alembert's principle with the principle of virtual velocities, or virtual work, Lagrange produced the first successful one, and he made it the foundation of his very famous *Mécanique Analytique*. The second one, the principle of least action, was proposed by Maupertuis in 1740, but was not firmly established before the time of Jacobi, and even then only in the domain of conservative forces. The third one is Hamilton's principle, and it seems to have grown out of the principle of least action. It is very widely known, and much used, although it is valid only when the constraints are integral, or if given in a differential form are integrable. The last one, Gauss' principle of least constraint, is valid under all circumstances, but does not seem to have been widely used.

In the discussion of these principles which follows, the application has been made to the motion of a single particle only, but it should be understood that each of these principles is equally applicable to systems of particles, with or without constraints.

I. D'ALEMBERT'S PRINCIPLE

383. The Principle of Virtual Work.—In the section on Statics (Sec. 165), it was pointed out that if a particle is in equilibrium under the action of any system of forces, the components of the resultant of which are X, Y, and Z, and if the particle be subjected to an arbitrary infinitesimal displacement, the components of which are δx, δy, and δz (a notation which is due

to Lagrange), then the work done in this displacement is zero. This is what is called *virtual work.* This principle of equilibrium can be expressed by the equation (Sec. 58)

$$X\delta x + Y\delta y + Z\delta z = 0.$$

At the time of d'Alembert (1717 to 1783), two classes of forces were recognized, living forces (*vis viva*) and dead forces (*vis mortua*). Living forces produced an actual acceleration in a particle, while dead forces did not. Thus, for a particle in equilibrium, all the forces involved are dead. D'Alembert called the product of mass times acceleration the *effective force* (also commonly called the *force of inertia*), its components being mx'', my'', and mz''. D'Alembert's principle is this: If a particle is not in equilibrium, the work done by all the forces which are acting, both living and dead, in any arbitrary displacement is equal to the work done by the effective forces in the same arbitrary displacement; or, expressed in an equation,

$$X\delta x + Y\delta y + Z\delta z = mx''\delta x + my''\delta y + mz''\delta z.$$

If the right member is transferred to the left side, this equation becomes

$$(X - mx'')\delta x + (Y - my'')\delta y + (Z - mz'')\delta z = 0, \tag{1}$$

which, expressed in words, states that if to the forces which are actually acting upon the particle the effective forces, reversed in direction, are added, the particle is in equilibrium. By this means, the problems of both statics and dynamics are reduced to the single principle of virtual work, equilibrium of the forces always existing.

This principle was developed by d'Alembert in his *Traité de Dynamique* published in 1743. It was reduced to its general analytic form, as given above, by Lagrange, who made it, rather than Newton's laws, the basis of his famous *Mécanique Analytique*, published in 1788. In this work a generalization of Eq. (1), or rather, an extension of Eq. (1) to all of the particles of a system, was made to serve as a starting point for the solution of any problem whatever in dynamics. Lagrange's work has excited universal admiration. On account of its unity and the elegance and beauty of its development, Sir William Hamilton called it "a kind of scientific poem." This is a tribute to Lagrange, however, rather than to his subject, for with respect to beauty of mathematical form Lagrange is the Shakespeare of mathematicians.

It is not necessary that the variations be expressed in rectangular coordinates. If another system, in which the coordinates are q_1, q_2, and q_3, is chosen, and if

$$x = x(q_1, q_2, q_3; t), \qquad y = y(q_1, q_2, q_3; t), \qquad z = z(q_1, q_2, q_3; t)$$

express the relations between the two systems, then

$$\left.\begin{aligned} \delta x &= \frac{\partial x}{\partial q_1}\delta q_1 + \frac{\partial x}{\partial q_2}\delta q_2 + \frac{\partial x}{\partial q_3}\delta q_3, \\ \delta y &= \frac{\partial y}{\partial q_1}\delta q_1 + \frac{\partial y}{\partial q_2}\delta q_2 + \frac{\partial y}{\partial q_3}\delta q_3, \\ \delta z &= \frac{\partial z}{\partial q_1}\delta q_1 + \frac{\partial z}{\partial q_2}\delta q_2 + \frac{\partial z}{\partial q_3}\delta q_3; \end{aligned}\right\} \tag{2}$$

and the variations of q_1, q_2, q_3 can be regarded as arbitrary, and those of x, y, z as dependent. Inasmuch as these arbitrary variations have no relation to the actual motion of the particle except that they must be compatible with the constraints, if there are any, the time t plays the rôle of a mere constant in making them. If there is one constraint, and the q's are properly chosen, the variation of q_3 will be zero; and if there are two constraints, the variations of q_2 and q_3 will be zero, and the motion is along a given curve.

If Eq. (2) is substituted in Eq. (1), there results

$$\begin{aligned} &\left[(X - mx'')\frac{\partial x}{\partial q_1} + (Y - my'')\frac{\partial y}{\partial q_1} + (Z - mz'')\frac{\partial z}{\partial q_1}\right]\delta q_1 \\ &+ \left[(X - mx'')\frac{\partial x}{\partial q_2} + (Y - my'')\frac{\partial y}{\partial q_2} + (Z - mz'')\frac{\partial z}{\partial q_2}\right]\delta q_2 \\ &+ \left[(X - mx'')\frac{\partial x}{\partial q_3} + (Y - my'')\frac{\partial y}{\partial q_3} + (Z - mz'')\frac{\partial z}{\partial q_3}\right]\delta q_3 = 0. \end{aligned}$$

Since the variations of the q's are arbitrary, their coefficients are zero. On equating these coefficients to zero, Eq. (3) of Sec. 347 results. It was in this manner that Lagrange derived his equations from d'Alembert's principle.

II. THE PRINCIPLE OF LEAST ACTION

384. Historical.—Influenced by certain metaphysical or theological considerations, Maupertuis stated in 1740 that it was reasonable to suppose that the activities of nature are conducted with the least possible effort and that the integral

$$\int mvds = \text{action}$$

is always a minimum; m being the mass of a particle, v its speed, and ds an element of the curve which the particle describes; the integral being taken along the curve between any two points of the curve. In 1744, Maupertuis published two memoirs in which he derived the laws of reflection and refraction of light, and of collisions of two bodies, by means of this principle, which he called the *principle of least action.*

In the same year, 1744, Euler published a proof, based upon purely mechanical considerations, that this integral for orbits described by a particle under the action of a central force was either a maximum or a minimum. Finally, in 1788 in his *Mécanique Analytique,* Lagrange showed that in general this integral is either a maximum or a minimum for any actual motion, provided the force acting is a conservative one, and provided also that the constraints, if any, are independent of the time; otherwise, the integral may be neither a maximum nor a minimum. Lagrange regarded the principle as stated by Maupertuis as vague and lacking in precision; nevertheless, he retained the name which Maupertuis had given it, the principle of least action.

In the *Vorlesungen über Dynamik,* Jacobi remarked that he could not understand the exposition of this principle by either Lagrange or Poisson, and constructed a new proof, thus for the first time establishing the principle upon a sound basis.

385. Extremals in the Calculus of Variations.—Let a particle of mass m, moving under the action of a force for which the potential function is U, describe the arc P_1P_2 (Fig. 183) denoted by the letter L. Since the force is a conservative one, the energy is constant and, therefore,

$$\frac{1}{2}mv^2 = U + H,$$

where H, a constant, is the total energy. Let L_1 be another curve joining the points A and B and lying everywhere infinitely near L. Let a second particle of mass m describe the curve L_1 in such a way that for its motion also

$$\frac{1}{2}mv^2 = U + H, \tag{1}$$

although, for such a motion, constraints would be necessary. It will be observed that for each point of the curve L_1 the speed of the particle is determined by Eq. (1), and therefore the time

required for describing the curve L_1 is not the same, in general, as that required for the curve L.

The definite integral

$$A = \int_{P_1}^{P_2} mvds, \tag{2}$$

taken along any curve between the points P_1 and P_2, is called the *action*, and the change in the action from a given curve to another one in its immediate neighborhood is called the *variation of the action*, and is written

$$\delta A = \delta \int_{P_1}^{P_2} mvds.$$

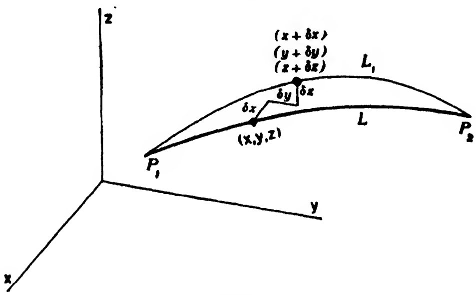


FIG. 183.

A curve which maximizes or minimizes a definite integral is called an *extremal* in the calculus of variations; and the variation of the definite integral for an extremal is zero, as would be expected from the general principles of maxima and minima. The principle of least action then, as formulated by Jacobi, is that the arc, which is described naturally between two given points by a particle which is acted upon by a force, which is derived from a potential function U which does not contain the time, is an extremal for the action, subject to the condition that the energy is a given constant along every curve. Hence, the curve which is followed by the particle is such that

$$\delta A = \delta \int_{P_1}^{P_2} mvds = 0.$$

386. Jacobi's Proof of the Principle of Least Action.—From the energy integral (Eq. 385.1), there is derived:

$$mv = \sqrt{2m(U + H)}. \tag{1}$$

Therefore, the action

$$A = \int_{P_1}^{P_2} \sqrt{2m(U + H)}ds,$$

under the conditions which are given, is purely geometric in character, the time having been eliminated altogether.

The x-, y-, and z-coordinates of the particle which describes the curve L can be regarded as functions of a parameter λ; say,

$$x = \varphi(\lambda), \qquad y = \psi(\lambda), \qquad z = \omega(\lambda).$$

Let λ_1 and λ_2 be two values of the parameter which correspond to the points P_1 and P_2, so that the coordinates of P_1 are $\varphi(\lambda_1)$, $\psi(\lambda_1)$, and $\omega(\lambda_1)$, and the coordinates of P_2 are $\varphi(\lambda_2)$, $\psi(\lambda_2)$, and $\omega(\lambda_2)$. The equations of any neighboring curve L_1, which also passes through P_1 and P_2, can be written

$$\begin{aligned} x &= \varphi(\lambda) + \epsilon(\lambda - \lambda_1)(\lambda_2 - \lambda)\varphi_1(\lambda), \\ y &= \psi(\lambda) + \epsilon(\lambda - \lambda_1)(\lambda_2 - \lambda)\psi_1(\lambda), \\ z &= \omega(\lambda) + \epsilon(\lambda - \lambda_1)(\lambda_2 - \lambda)\omega_1(\lambda); \end{aligned}$$

and the variations of x, y, z are then

$$\left.\begin{aligned} \delta x &= \epsilon(\lambda - \lambda_1)(\lambda_2 - \lambda)\varphi_1(\lambda), \\ \delta y &= \epsilon(\lambda - \lambda_1)(\lambda_2 - \lambda)\psi_1(\lambda), \\ \delta z &= \epsilon(\lambda - \lambda_1)(\lambda_2 - \lambda)\omega_1(\lambda); \end{aligned}\right\} \tag{2}$$

where ϵ is an infinitesimal parameter and φ_1, ψ_1, and ω_1 are functions of λ of such a nature that δx, δy, and δz are continuous for $\lambda_1 \leqq \lambda \leqq \lambda_2$ and vanish for λ equal to λ_1 and for λ equal to λ_2.

For the simplification of notation, let derivatives with respect to λ be denoted by subscripts, so that

$$\frac{dx}{d\lambda} = x_\lambda, \qquad \frac{dy}{d\lambda} = y_\lambda, \qquad \frac{dz}{d\lambda} = z_\lambda.$$

Then the expression for the arc element ds becomes

$$ds = \sqrt{x_\lambda^2 + y_\lambda^2 + z_\lambda^2}\, d\lambda; \tag{3}$$

and the condition that the curve L shall be an extremal is that

$$\delta A = \delta \int_{\lambda_1}^{\lambda_2} \sqrt{2m(U + H)}\, \sqrt{x_\lambda^2 + y_\lambda^2 + z_\lambda^2}\, d\lambda = 0. \tag{4}$$

Again, for the simplification of notation, let

$$\Lambda = \frac{m\sqrt{x_\lambda^2 + y_\lambda^2 + z_\lambda^2}}{\sqrt{2m(U + H)}}.$$

Then, on developing the variations, Eq. (4) becomes

$$\delta A = 0 = \int_{\lambda_1}^{\lambda_2} \Lambda\left(\frac{\partial U}{\partial x}\delta x + \frac{\partial U}{\partial y}\delta y + \frac{\partial U}{\partial z}\delta z\right)d\lambda$$
$$+ \int_{\lambda_1}^{\lambda_2} \frac{m}{\Lambda}(x_\lambda \delta x_\lambda + y_\lambda \delta y_\lambda + z_\lambda \delta z_\lambda)d\lambda. \quad (5)$$

It is evident from Eq. (2) that

$$\delta x_\lambda = \delta\frac{dx}{d\lambda} = \frac{d(\delta x)}{d\lambda},$$

and, therefore,

$$\delta x_\lambda d\lambda = d(\delta x); \quad (6)$$

and similarly with the other coordinates. On substituting Eq. (6) in the second integral of Eq. (5) and then integrating by parts, this second integral is seen to be equal to

$$\left[\frac{m}{\Lambda}(x_\lambda \delta x + y_\lambda \delta y + z_\lambda \delta z)\right]_{\lambda_1}^{\lambda_2}$$
$$- \int_{\lambda_1}^{\lambda_2} m\left[\delta x \cdot d\left(\frac{x_\lambda}{\Lambda}\right) + \delta y \cdot d\left(\frac{y_\lambda}{\Lambda}\right) + \delta z \cdot d\left(\frac{z_\lambda}{\Lambda}\right)\right]. \quad (7)$$

The integrated part of Eq. (7) vanishes, since the variations δx, δy, and δz vanish at both limits. If the integral in Eq. (7) is combined with the first integral in Eq. (5), it is found that

$$\delta A = 0 = \int_{\lambda_1}^{\lambda_2} \left\{\left[\frac{\partial U}{\partial x}\Lambda d\lambda - md\left(\frac{x_\lambda}{\Lambda}\right)\right]\delta x\right.$$
$$\left. + \left[\frac{\partial U}{\partial y}\Lambda d\lambda - md\left(\frac{y_\lambda}{\Lambda}\right)\right]\delta y + \left[\frac{\partial U}{\partial z}\Lambda d\lambda - md\left(\frac{z_\lambda}{\Lambda}\right)\right]\delta z\right\}. \quad (8)$$

Since this integral must vanish for every system of variations δx, δy, and δz, it is necessary that the coefficient of each variation separately shall be zero. Therefore,

$$\left.\begin{aligned} md\left(\frac{x_\lambda}{\Lambda}\right) &= \frac{\partial U}{\partial x}\Lambda d\lambda, \\ md\left(\frac{y_\lambda}{\Lambda}\right) &= \frac{\partial U}{\partial y}\Lambda d\lambda, \\ md\left(\frac{z_\lambda}{\Lambda}\right) &= \frac{\partial U}{\partial z}\Lambda d\lambda. \end{aligned}\right\} \quad (9)$$

Now, by virtue of Eq. (1),

$$m\frac{ds}{dt} = \sqrt{2m(U + H)},$$

and by Eq. (3)

$$ds = \sqrt{x_\lambda^2 + y_\lambda^2 + z_\lambda^2}\,d\lambda;$$

hence,

$$dt = \frac{m\sqrt{x_\lambda^2 + y_\lambda^2 + z_\lambda^2}}{\sqrt{2m(U + h)}}d\lambda = \Lambda d\lambda, \tag{10}$$

and

$$\frac{x_\lambda}{\Lambda} = x', \qquad \frac{y_\lambda}{\Lambda} = y', \qquad \frac{z_\lambda}{\Lambda} = z'.$$

Equations (9) now reduce to

$$mx'' = \frac{\partial U}{\partial x}, \qquad my'' = \frac{\partial U}{\partial y}, \qquad mz'' = \frac{\partial U}{\partial z}, \tag{11}$$

which are the equations of motion in rectangular coordinates.

The equations of motion, therefore, can be derived from the principle of least action. The converse also is true; for, given the equations of motion and the energy integral, it is possible to go backward from Eq. (11) through Eqs. (10) and (9) to Eq. (8), and show that the curve described naturally is an extremal of A. The curves do not minimize the action in general, but Jacobi showed that they do minimize it if the arc P_1P_2 is sufficiently short.

III. HAMILTON'S PRINCIPLE

387. Proof of Hamilton's Principle.—Hamilton's principle is strikingly similar to the principle of least action. There is scarcely any doubt that Hamilton encountered the same difficulty in understanding Lagrange's discussion of the principle of least action that Jacobi did, but instead of developing a satisfactory proof for the principle of least action, he branched off from Lagrange's discussion and developed a new principle, which is more comprehensive than that of least action in that it does not presuppose the existence of an energy integral; nor does it require that the constraints, if any, shall be free from the time. Appell, however, has shown that it does require that the constraints, if any, shall be independent of the velocities.

Equation (1) of Sec. 383 can be written

$$-m(x''\delta x + y''\delta y + z''\delta z) + (X\delta x + Y\delta y + Z\delta z) = 0. \tag{1}$$

Let L (Fig. 183) be the curve described naturally by the particle between any two points P_1 and P_2, and let L_1 be any neighboring curve through P_1 and P_2 which is described by a second particle *in the same time* that is required for the curve L.

Then

$$\delta x = \epsilon(t - t_1)(t_2 - t)\varphi_1(t),$$
$$\delta y = \epsilon(t - t_1)(t_2 - t)\varphi_2(t),$$
$$\delta z = \epsilon(t - t_1)(t_2 - t)\varphi_3(t),$$

where φ_1, φ_2, φ_3 are arbitrary functions of t subject to the condition that the variations are continuous and vanish at t_1 and t_2, which are the values of t which correspond to the points P_1 and P_2. From these definitions, it is evident that $\delta x'$ is the same thing as $(\delta x)'$. This being true, it is easily seen that

$$x''\delta x = (x'\delta x)' - x'(\delta x)',$$
$$= (x'\delta x)' - \frac{1}{2}\delta\, x'^2;$$

similarly,

$$y''\delta y = (y'\delta y)' - \frac{1}{2}\delta\, y'^2,$$

and

$$z''\delta z = (z'\delta z)' - \frac{1}{2}\delta\, z'^2,$$

so that Eq. (1) becomes

$$-m(x'\delta x + y'\delta y + z'\delta z)' + \frac{1}{2}m\delta(x'^2 + y'^2 + z'^2) + (X\delta x + Y\delta y + Z\delta z) = 0. \quad (2)$$

If Eq. (2) is multiplied by dt and integrated from t_1 to t_2, the first term of the result

$$-m\Big[x'\delta x + y'\delta y + z'\delta z\Big]_{t_1}^{t_2}$$

vanishes, since the variations δx, δy, δz are zero at both limits. On setting

$$T = \frac{1}{2}m(x'^2 + y'^2 + z'^2),$$

the integral of Eq. (2) can be written

$$\int_{t_1}^{t_2}[\delta T + X\delta x + Y\delta y + Z\delta z]dt = 0. \quad (3)$$

This is known as Hamilton's principle. Its expression can be simplified somewhat if there exists a force function U such that

$$\delta U = X\delta x + Y\delta y + Z\delta z,$$

for then Eq. (3) becomes

$$\int_{t_1}^{t_2}(\delta T + \delta U)dt = \delta\int_{t_1}^{t_2}(T + U)dt = 0. \quad (4)$$

This is the form in which it is usually encountered.

Since the potential function U is the negative of the potential energy V, and since

$$\frac{1}{t_2 - t_1}\int_{t_1}^{t_2}(T - V)\,dt$$

is the average value with respect to the time of the difference between the kinetic and potential energies, it is seen that Hamilton's principle asserts that in the motion which actually occurs between any two given points the time average of the difference between the kinetic and potential energies is either a maximum or a minimum when compared with any other infinitely near motion between the same two points provided the time interval and the potential function are the same for both motions.

388. Equations of Lagrange Derived from Hamilton's Principle.—It is evident that Hamilton's principle is independent of any particular coordinate system, since T is the kinetic energy while X, Y, and Z are the components of the force F which is acting upon the particle, so that the expression

$$X\delta x + Y\delta y + Z\delta z = \mathbf{F}\cdot\Delta,$$

or the work done by $\mathbf{F}$ in any infinitesimal displacement Δ.

Let T, $\mathbf{F}$, Δ be expressed in any desired system for which the coordinates are q_1, q_2, and q_3. Then

$$\delta T = \sum_{i=1}^{3}\frac{\delta T}{\delta q_i}\delta q_i + \sum_{i=1}^{3}\frac{\delta T}{\delta q_i'}\delta q_i'.$$

From the discussion in Sec. 387, it is evident that

$$\delta q_i' = d\frac{\delta q_i}{dt}, \qquad \delta q_i'\cdot dt = d(\delta q_i),$$

so that

$$\int_{t_1}^{t_2}\delta T dt = \int_{t_1}^{t_2}\sum_{i=1}^{3}\frac{\delta T}{\delta q_i}\delta q_i dt + \int_{P_1}^{P_2}\sum_{i=1}^{3}\frac{\delta T}{\delta q_i'}d(\delta q_i). \tag{1}$$

On integrating the last integral of Eq. (1) by parts, it is found that

$$\int_{P_1}^{P_2}\sum_{i=1}^{3}\frac{\partial T}{\partial q_i'}d(\delta q_i) = \left[\sum_{i=1}^{3}\frac{\partial T}{\partial q_i'}\delta q_i\right]_{P_1}^{P_2} - \int_{t_1}^{t_2}\sum_{i=1}^{3}\frac{d}{dt}\left(\frac{\partial T}{\partial q_i'}\right)\delta q_i dt. \tag{2}$$

Since the variations δq_i vanish at both limits, the integrated part of Eq. (2) is zero, and therefore Eq. (1) becomes

$$\int_{t_1}^{t_2}\delta T dt = \int_{t_1}^{t_2}\sum_{i=1}^{3}\left[-\left(\frac{\partial T}{\partial q_i'}\right)' + \frac{\partial T}{\partial q_i}\right]\delta q_i\cdot dt.$$

If Q_1, Q_2, and Q_3 are the components of **F** in the directions q_1, q_2, and q_3, and δq_1, δq_2, and δq_3 are the components of Δ, then

$$X\delta x + Y\delta y + Z\delta z = Q_1\delta q_1 + Q_2\delta q_2 + Q_3\delta q_3,$$

for they represent the work done in the infinitesimal displacement in the two systems of coordinates. Hence, the expression (Eq. (387.3)) for Hamilton's principle becomes

$$\int_{t_1}^{t_2}\sum_{i=1}^{3}\left[-\left(\frac{\partial T}{\partial q_i'}\right)' + \frac{\partial T}{\partial q_i} + Q_i\right]\delta q_i dt = 0;$$

and since this expression vanishes whatever the variations δq_i may be, the coefficient of each δq_i separately is zero. Hence,

$$\left(\frac{\partial T}{\partial q_i'}\right)' - \frac{\partial T}{\partial q_i} = Q_i \qquad i = 1, 2, 3. \tag{3}$$

These are the equations of Lagrange.

If the time occurs explicitly in any of these expressions, it is, of course, to be regarded as a constant in making the variations.

IV. GAUSS' PRINCIPLE OF LEAST CONSTRAINT

389. Statement and Proof of Gauss' Principle.—Gauss' principle of least constraint adds nothing to the knowledge of the motion of a free particle. Its purpose is to compare the position of a constrained particle with the position which the particle would have had if it had been free during the interval of time dt immediately preceding the instant of comparison. The difference between the two positions evidently is due to the constraints, and Gauss' principle asserts that of all possible geometrical displacements which are compatible with the constraints the smallest one is the one which actually occurs dynamically.

Suppose the particle of mass m is at the point O at the instant t. It is acted upon by a system of forces which has a resultant F, of which the components are X, Y, and Z; the resultant includes the friction, if there is any, but does not include the constraints. The coordinates of the particle at O are x, y, and z; its components of velocity are x', y', and z'; and its components of acceleration are x'', y'', and z''; so that, in accordance with Tay-

lor's theorem, its positional coordinates at the instant $t + dt$ are

$$\text{(actual)}\qquad \begin{aligned} a_x &= x + x'dt + \frac{1}{2}x''dt^2 + \cdots, \\ a_y &= y + y'dt + \frac{1}{2}y''dt^2 + \cdots, \\ a_z &= z + z'dt + \frac{1}{2}z''dt^2 + \cdots, \end{aligned}$$

the terms of higher degree having no value for the purpose which is under consideration.

Suppose, again, that the constraints had been released abruptly at the instant t, everything else remaining the same. Then the positional coordinates at the instant $t + dt$ would have been

$$\text{(hypothetical)}\qquad \begin{aligned} h_x &= x + x'dt + \frac{1}{2}\frac{X}{m}dt^2 + \cdots, \\ h_y &= y + y'dt + \frac{1}{2}\frac{Y}{m}dt^2 + \cdots, \\ h_z &= z + z'dt + \frac{1}{2}\frac{Z}{m}dt^2 + \cdots. \end{aligned}$$

Let these two points be denoted by a and h, respectively. Then the projections of the line $\overline{ah}$ upon the axes of reference are

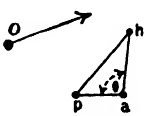


FIG. 184.

$$\text{(difference)}\qquad \begin{aligned} &\frac{1}{2}\left(x'' - \frac{X}{m}\right)dt^2, \\ &\frac{1}{2}\left(y'' - \frac{Y}{m}\right)dt^2, \\ &\frac{1}{2}\left(z'' - \frac{Z}{m}\right)dt^2. \end{aligned} \tag{1}$$

Multiplied by m, these expressions are proportional to the components of the force due to the constraints. By d'Alembert's principle (Eq. (383.1)),

$$(X - mx'')\delta x + (Y - my'')\delta y + (Z - mz'')\delta z = 0 \tag{2}$$

for every displacement δx, δy, δz which is compatible with the constraints.

Let p be any point which is compatible with the constraints and which is infinitely near a. Then $\overline{ap}$ is a displacement which is compatible with the constraints. From Eqs. (1) and (2), it is seen that the work done in this displacement is the work done

by a force which is proportional to the vector $\overline{ha}$ and, since the work done is zero, the factor of proportionality is immaterial. If θ is the angle, therefore, between the force and the displacement,

$$\overline{ha} \cdot \overline{ap} \cos \theta = 0.$$

Therefore, $\theta = \pi/2$, and $\overline{ha}$ is one side of a right triangle of which $\overline{hp}$ is the hypothenuse. It is therefore shorter than $\overline{hp}$, which is any other geometrically possible constrained displacement, and is therefore perpendicular to the constraining line or surface. Thus, Gauss' principle is proved.

390. Analytic Formulation of Gauss' Principle.—The components of $\overline{ha}$ are, according to Eq. (389.1),

$$\frac{1}{2}(mx'' - X)\frac{dt^2}{m}, \qquad \frac{1}{2}(my'' - Y)\frac{dt^2}{m}, \qquad \frac{1}{2}(mz'' - Z)\frac{dt^2}{m}.$$

Hence,

$$2\frac{m^2}{dt^4}\overline{ha}^2 = \frac{1}{2}[(mx'' - X)^2 + (my'' - Y)^2 + (mz'' - Z)^2]$$

is a minimum. This can be formulated in words as follows: *For a given force $F(X, Y, Z)$ and given constraints, the function*

$$\frac{1}{2}[(mx'' - X)^2 + (my'' - Y)^2 + (mz'' - Z)^2],$$

at every instant t, is smaller for the motion that actually occurs than for any other geometrically possible motion.

APPENDIX

NOTE ON THE HYPERBOLIC FUNCTIONS

It is assumed as known that the functions e^x, $\cos x$, $\sin x$ are expansible in powers of the argument x, and that these expansions

$$\left.\begin{aligned} e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots, \\ \cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots, \\ \sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots, \end{aligned}\right\} \tag{1}$$

are convergent for all values of the argument.

In the expression for the exponential, let x be replaced by $i\varphi$, where $i = \sqrt{-1}$. Then

$$\begin{aligned} e^{i\varphi} &= 1 + i\varphi + \frac{i^2\varphi^2}{2!} + \frac{i^3\varphi^3}{3!} + \frac{i^4\varphi^4}{4!} + \cdots, \\ &= 1 + i\varphi - \frac{\varphi^2}{2!} - i\frac{\varphi^3}{3!} + \frac{\varphi^4}{4!} + \cdots, \\ &= \left(1 - \frac{\varphi^2}{2!} + \frac{\varphi^4}{4!} - \cdots\right) + i\left(\varphi - \frac{\varphi^3}{3!} + \frac{\varphi^5}{5!} - \cdots\right); \end{aligned}$$

and therefore, since the expressions within the parentheses are the $\cos \varphi$ and $\sin \varphi$,

$$e^{i\varphi} = \cos \varphi + i \sin \varphi;$$

similarly,

$$e^{-i\varphi} = \cos \varphi - i \sin \varphi.$$

The sum and the difference of these two expressions give

$$\cos \varphi = \frac{e^{i\varphi} + e^{-i\varphi}}{2}, \qquad \sin \varphi = \frac{e^{i\varphi} - e^{-i\varphi}}{2i}. \tag{2}$$

Now let φ be replaced by $i\psi$. Then

$$\cos i\psi = \frac{e^{\psi} + e^{-\psi}}{2}, \qquad \sin i\psi = i\,\frac{e^{\psi} - e^{-\psi}}{2}. \tag{3}$$

The definitions of $\cosh \psi$ and $\sinh \psi$ are

$$\cosh \psi = \frac{e^{\psi} + e^{-\psi}}{2}, \qquad \sinh \psi = \frac{e^{\psi} - e^{-\psi}}{2}, \tag{4}$$

so that

$$\left.\begin{aligned} \cosh\psi &= 1 + \frac{\psi^2}{2!} + \frac{\psi^4}{4!} + \frac{\psi^6}{6!} + \cdots, \\ \sinh\psi &= \psi + \frac{\psi^3}{3!} + \frac{\psi^5}{5!} + \frac{\psi^7}{7!} + \cdots. \end{aligned}\right\} \tag{5}$$

A comparison of Eqs. (3) and (4) shows that

$$\cos i\psi = \cosh\psi, \qquad \sin i\psi = i\sinh\psi. \tag{6}$$

Also

$$\begin{aligned} \tan i\psi &= \frac{\sin i\psi}{\cos i\psi} = \frac{i\sinh\psi}{\cosh\psi} = +i\tanh\psi, \\ \sec i\psi &= \frac{1}{\cos i\psi} = \frac{1}{\cosh\psi} = \operatorname{sech}\psi, \\ \cot i\psi &= \frac{\cos i\psi}{\sin i\psi} = \frac{\cosh\psi}{i\sinh\psi} = -i\coth\psi, \\ \operatorname{cosec} i\psi &= \frac{1}{\sin i\psi} = \frac{1}{i\sinh\psi} = -i\operatorname{cosech}\psi. \end{aligned}$$

If the expressions in Eq. (2) for $\sin\varphi$ and $\cos\varphi$ are squared and then added, it is found that

$$\cos^2\varphi + \sin^2\varphi = 1. \tag{7}$$

In precisely the same way, it is found from Eq. (4) that

$$\cosh^2\varphi - \sinh^2\varphi = 1; \tag{8}$$

but Eq. (8) could have been derived from Eq. (7) by changing φ into $i\varphi$ and then applying Eq. (6).

On multiplying together the expressions for $\sin\varphi$ and $\cos\varphi$ in Eq. (2), it is found that

$$2\cos\varphi\sin\varphi = \frac{e^{2i\varphi} - e^{-2i\varphi}}{2i} = \sin 2\varphi;$$

and similarly from Eq. (4)

$$2\cosh\varphi\sinh\varphi = \frac{e^{2\varphi} - e^{-2\varphi}}{2} = \sinh 2\varphi.$$

Also,

$$\begin{aligned} \cos^2\varphi - \sin^2\varphi &= \frac{e^{2i\varphi} + e^{-2i\varphi}}{2} = \cos 2\varphi, \\ \cosh^2\varphi + \sinh^2\varphi &= \frac{e^{2\varphi} + e^{-2\varphi}}{2} = \cosh 2\varphi. \end{aligned}$$

A little experimenting of this kind will soon convince the student that all of the formulas of trigonometry can be derived from Eq. (2), and corresponding expressions for the hyperbolic functions can be derived from Eq. (4). Furthermore, all of the expressions for the hyperbolic functions can be derived from the

corresponding expressions for the trigonometric functions by changing φ into $i\varphi$ and then applying Eq. (6); and nothing more serious happens than an occasional change of sign.

By differentiating either Eq. (4) or Eq. (5), it is found that

$$\frac{d}{d\varphi}\cosh\varphi = +\sinh\varphi \qquad \frac{d}{d\varphi}\sinh\varphi = +\cosh\varphi.$$

No change of sign occurs, as for the corresponding trigonometric functions. Also,

$$\frac{d}{d\varphi}\tanh\varphi = \operatorname{sech}^2\varphi, \qquad \frac{d}{d\varphi}\coth\varphi = -\operatorname{cosech}^2\varphi,$$

$$\frac{d}{d\varphi}\operatorname{sech}\varphi = -\operatorname{sech}\varphi\tanh\varphi,$$

$$\frac{d}{d\varphi}\operatorname{cosech}\varphi = -\operatorname{cosech}\varphi\coth\varphi.$$

As for integration, it will be verified readily that

$$\int\frac{dx}{\sqrt{x^2+1}} = \sinh^{-1}x, \qquad \int\frac{dx}{\sqrt{x^2-1}} = \cosh^{-1}x,$$

$$\int\frac{dx}{1-x^2} = \tanh^{-1}x, \qquad \int\frac{dx}{x^2-1} = -\coth^{-1}x,$$

$$\int\frac{dx}{x\sqrt{1-x^2}} = -\operatorname{sech}^{-1}x, \qquad \int\frac{dx}{x\sqrt{1+x^2}} = -\operatorname{cosech}^{-1}x.$$

By setting

$$x = a\cos\varphi, \qquad y = a\sin\varphi,$$

a parametric representation of a circle is obtained, since

$$x^2 + y^2 = a^2.$$

Similarly, by setting

$$x = a\cosh\varphi, \qquad y = a\sinh\varphi,$$

a parametric representation of an equilateral hyperbola is obtained, since

$$x^2 - y^2 = a^2.$$

Thus the hyperbolic functions are related to the equilateral hyperbola in much the same way that the trigonometric functions are related to the circle. (See also Sec. 299.)

The following is a brief table of the hyperbolic functions:

No.	sinh	cosh	tanh	coth	sech	csch
0.0	0.00	1.00	0.00	∞	1.00	∞
0.1	0.10	1.01	0.10	10.0	0.99	10.0
0.2	0.20	1.02	0.20	5.07	0.98	5.00
0.3	0.30	1.05	0.29	3.43	0.95	3.33
0.4	0.41	1.08	0.38	2.63	0.92	2.44
0.5	0.52	1.13	0.46	2.16	0.88	1.92
0.6	0.64	1.19	0.54	1.86	0.84	1.56
0.7	0.76	1.26	0.60	1.65	0.79	1.32
0.8	0.89	1.34	0.66	1.51	0.75	1.12
0.9	1.03	1.43	0.72	1.40	0.70	0.97
1.0	1.18	1.54	0.76	1.31	0.65	0.85
1.5	2.13	2.35	0.91	1.10	0.43	0.47
2.0	3.63	3.76	0.96	1.04	0.27	0.28
2.5	6.05	6.13	0.99	1.01	0.16	0.17
3.0	10.0	10 1	1.00	1.00	0.10	0.10
4.0	27.3	27.3	1.00	1.00	0.04	0.04
5.0	74.2	74.2	1.00	1.00	0.01	0.01

Miscellaneous Constants

1 inch = 2.540005 centimeters.
1 foot = 30.480061 centimeters.
1 mile = 1.609347 kilometers.
1 meter = 39.37 inches.
1 pound (mass) = 453.59243 grams.
1 pound (weight) = 444,820 dynes.
1 kilogram = 2.204622 pounds.
1 poundal = 13,825 dynes.
1 foot-pound = 13,558,200 ergs.
1 calorie (mean) = 4.186×10^7 ergs.
1 cubic foot water = 62.4 pounds.
1 cubic foot air = 0.0806 pound at 0°C., bar. 30″.
1 knot = 6080.2 feet (= 1 nautical mile), per hour.
Equatorial radius of the earth = 3963.34 miles.
Polar radius of the earth = 3950.00 miles.
Radius of sphere of equal volume = 3958.89 miles.
Mean density of earth = 5.527
Mass of sun = 332,000 times mass of earth.

Mass of moon = 0.01227 times mass of earth.
Mean distance of sun = 92,800,000 miles.
Mean distance of moon = 238,857 miles.
Velocity of light = 186,284 miles per second.
Gravitational constant = 6.658×10^{-8} cm.^{3} $g.^{-1}$ sec.$^{-2}$.
$g = 32.174 - 0.085 \cos 2l$.

$\pi = 3.14159265$	$\log \pi = 0.49714987$.
$e = 2.71828183$	$\log e = 0.43429448$.
radian = $57.^\circ 29578$	log = 1.75812263.

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